ALGEBRAIC STRUCTURES ON THE MODULI SPACE OF CURVES FROM REPRESENTATIONS OF VERTEX OPERATOR ALGEBRAS

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ABSTRACT. These are notes with my lectures at the University of Massachusetts, Amherst summer school on the Physical Mathematics of Quantum Field Theory.

INTRODUCTION

The moduli space $\overline{\mathrm{M}}_{g,n}$ of n-pointed (Deligne-Mumford) stable curves of genus g provides a natural environment in which we may study smooth curves and their degenerations. These spaces, for different values of g and n, are related to each other through systems of tautological maps. Algebraic structures on $\overline{\mathrm{M}}_{g,n}$ often reflect this, being governed by recursions, and amenable to inductive arguments. For these and other reasons, such structures are generally easier to work with. Sheaves on the moduli space of curves given by representations of vertex operator algebras (VOAs for short) exemplify this.

VOAs generalize commutative associative algebras as well as Lie algebras, and have played deep and important roles in both mathematics and mathematical physics. For instance, in understanding conformal field theories, finite group theory, and in the construction of knot invariants and 3-manifold invariants. Given nice enough VOAs and categories from which modules are selected, sheaves of coinvariants behave functorially with respect to these tautological maps.

In the first lecture, I will introduce the moduli spaces of curves that are involved in the construction, and also some of the questions that the sheaves may help answer. In lecture two I will introduce vertex operator algebras, and their modules giving some examples. In lecture three I will describe the sheaves of coinvariants and dual sheaves of conformal blocks, describing some of their important features. In the last lecture I will discuss a number of open problems.

1. LECTURE 1: THE MODULI SPACE OF CURVES AND VERTEX OPERATOR ALGEBRAS

A moduli space is a variety (or a scheme or a stack) that parametrizes some class of objects. The general moduli/parameter spaces philosophy goes something like the following:

- Objects X (like varieties with properties in common) can often correspond points in a moduli space $\mathcal{M}$. By studying $\mathcal{M}$ one can learn about X.
- Points $[X] \in \mathcal{M}$ with *good* properties often form a large (dense) open subset of $\mathcal{M}$.
- Points $[X] \in \mathcal{M}$ that don't have good properties occupy closed (proper) subsets of $\mathcal{M}$. The worse these points are, the smaller their ambient environment.

Today we will apply this philosophy to $\overline{\mathrm{M}}_g$, the moduli space of n-pointed Deligne-Mumford stable curves of genus $g \geq 2$. One dimensional algebraic varieties, arguably the simplest objects one studies, can be better understood as points on moduli spaces of curves. As curves arise in many contexts, moduli of curves are meeting grounds where a variety of techniques are applied in concert. In algebraic geometry, moduli of curves are particularly important: they help us understand smooth curves and their degenerations, and as special varieties, they have been one of the chief concrete, nontrivial settings where the nuanced theory of the minimal model program has been exhibited and explored [HH09, HH13, AFSvdW17, AFS17a, AFS17b]. They have also played a principal role as a prototype for moduli of higher dimensional varieties [KSB88, Ale02, HM06, HKT06, HKT09, CGK09].

It is not uncommon to refer to certain varieties as combinatorial: these include toric varieties: like projective space, weighted projective spaces, and certain blowups of those, Grassmannian varieties, or even more generally homogeneous varieties. These all come with group actions, and combinatorial data encoded in convex bodies keeps track of their important geometric features. Certain varieties like moduli of curves, have combinatorial structures reminiscent of varieties that are more traditionally considered to be combinatorial. As a result, various analogies have been made between them and the moduli of curves. Such comparisons have led to questions and conjectures, surprising formulas, and even arguments that have been used to detect and to prove some of the most important and often subtle geometric properties of the moduli space of curves.

As we shall see today, by looking at loci of curves with singularities, we are led to the study of moduli spaces $\overline{\mathrm{M}}_{g,n}$, parametrizing stable n-pointed curves of genus g. We'll also see that these spaces, for different g and n, are connected through tautological clutching and projection morphisms, give the system and these spaces a rich combinatorial structure. Algebraic structures on $\overline{\mathrm{M}}_{g,n}$ reflect this, and are often governed by recursions, and amenable to inductive arguments. Consequently, many questions can be reduced to moduli spaces of curves of smaller genus and fewer marked points. Problems about curves of positive genus can often be reduced to the smooth, projective, rational variety $\overline{\mathrm{M}}_{0,n}$, which can be constructed in a simple manner as a sequence of blowups of projective spaces. Today we will talk about this.

As you can see from other more complete surveys [Har84, Far09, Abr13, Coş17], this is a long studied subject with many points of focus!

1.1. **Why moduli?** The basic objects of study in algebraic geometry are varieties (or schemes or stacks). Zero sets of polynomials give algebraic varieties. The simplest are lines, which as can be seen in the picture below, taken together form varieties:

When learning about algebraic geometry, one typically starts with affine varieties, which in their simplest form are the zero sets of polynomials in some number of variables. Soon we learn that it is useful to homogenize those polynomials so we can study projective varieties for which there is more theory available. For instance, zero sets of degree d polynomials define curves in the affine plane, and homogeneous polynomials of degree d in three variables determine curves in $\mathbb{P}^2$, which

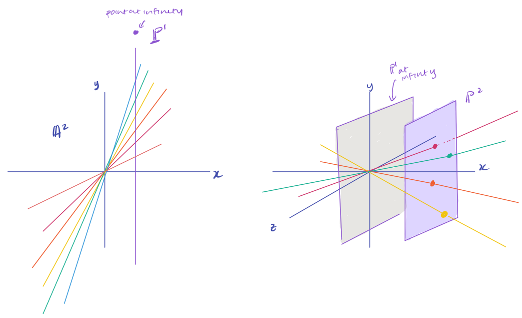


FIGURE 1. Imagining projective lines and spaces.

we can classify according to their genus

$$g = \frac{d(d-1)}{2}.$$

The genus of a curve is an invariant: If two curves have different genera, they can't be isomorphic. As some of you will discuss in the problem sessions, there are more geometric ways to define this number. For instance, the genus of a smooth curve C is

$$g = \dim \mathrm{H}^0(C, \omega_C) = \dim \mathrm{H}^1(C, \mathcal{O}_C),$$

where ω_C is the sheaf of regular 1-forms on C. If defined over the field of complex numbers, we may consider C as a Riemann surface, and the algebraic definition of genus is the same as the topological definition.

FIGURE 2. Picturing genus.

The simplest examples of plane curves have genus zero. These can be obtained as zero sets of conics in two variables:

$$f_{\alpha_\bullet}(x_1, x_2) = \sum_{j,k \geq 0, j+k \leq 2} \alpha_{jk} x_1^j x_2^k.$$

or as homogeneous polynomials of degree 2:

$$F_{a_\bullet}(x_0 : x_1 : x_2) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=2}} a_{ijk} x_0^i x_1^j x_2^k.$$

Note that the element

$$a_\bullet = [a_{200} : a_{110} : a_{101} : a_{020} : a_{011} : a_{002}] \in \mathbf{P}^5$$

determines the zero set $Z(F_{a_\bullet}) \subset \mathbf{P}^2$. In other words, there is a 5 dimensional family of rational curves. If we ask for only those curves that pass through a fixed set of points, say

$$p^1 = [1:0:0],\ p^2 = [0:1:0],\ p^3 = [0:0:1],\ \text{ and }\ p^4 = [1:1:1],$$

then since every point imposes a linear condition on the coefficients, we obtain a one dimensional family of 4-pointed rational curves.

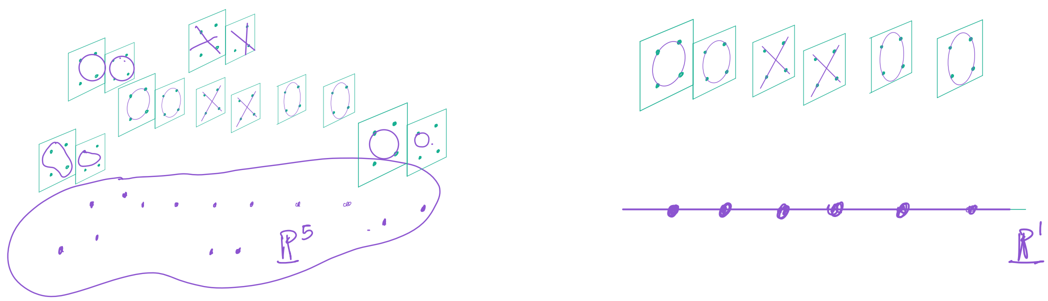


FIGURE 3. Families of 4-pointed rational curves.

Plane curves of genus 1 can be obtained as zero sets of cubic polynomials, and we can write down the general curve of genus 2 using the equation:

$$x_2^2 = x_1^6 + a_5x_1^5 + a_4x_1^4 + \cdots + a_1x_1 + a_0.$$

In other words, a point $(a_0, \ldots, a_5) \in \mathbf{A}^6$ determines a curve of genus 2, and there is a family of curves parametrized by an open subset of $\mathbf{A}^6$ that includes the general smooth curve of genus 2. As the coefficients change, the curves will sometimes have singularities.

1.2. **Moduli of curves.**

Definition 1.1. M_g *is the moduli space of smooth curves of genus g, the variety whose points are in one-to-one correspondence with isomorphism classes of smooth curves of genus* $g \geq 2$.

As smooth curves degenerate to curves with singularities, even if we just care about families of smooth curves it is useful to work with a a compactification of M_g – a proper space that contains M_g as a (dense) open subset. Such a space will necessarily parametrize curves with singularities.

We will consider the compactification $\overline{\mathrm{M}}_g$ whose points correspond to Deligne-Mumford stable curves of genus g. There are a number of choices of compactifications of M_g, and some of these receive birational morphisms from $\overline{\mathrm{M}}_g$ while others just receive rational maps from $\overline{\mathrm{M}}_g$. A few examples are given in the Appendix.

Definition 1.2. *A* ***stable curve*** C *of (arithmetic) genus g is a reduced, connected, one dimensional scheme such that*

(1) C has only ordinary double points as singularities.
(2) C has only a finite number of automorphisms.

Remark 1.1. *That C has finitely many automorphisms comes down to two conditions: (1) if C_i is a nonsingular rational component, then C_i meets the rest of the curve in at least three points, and (2) if C_i is a component of genus one, then it meets the rest of the curve in at least one point.*

Definition 1.3. $\overline{\mathrm{M}}_g$ *is the moduli space of stable curves of genus g, the variety whose points are in one-to-one correspondence with isomorphism classes of stable curves of genus* $g \geq 2$.

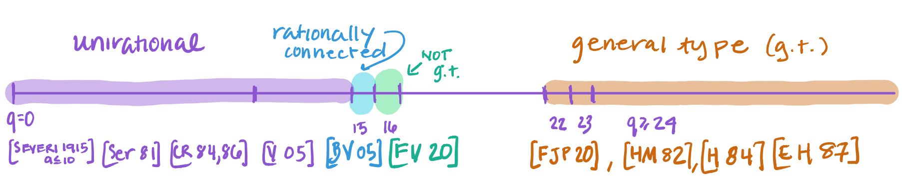
That such a variety $\overline{\mathrm{M}}_g$ exists is nontrivial. This was proved by Deligne and Mumford who constructed $\overline{\mathrm{M}}_g$ using Geometric Invariant Theory [DM69]. In the second lecture we will see Keel's construction of the space $\overline{\mathrm{M}}_{0,n}$.

This variety $\overline{\mathrm{M}}_g$ has the essential property that given any flat family $\mathcal{F} \to B$ of curves of genus g, there is a morphism $B \to \overline{\mathrm{M}}_g$, that takes a point $b \in B$ to the isomorphism class $[\mathcal{F}_b] \in \overline{\mathrm{M}}_g$ represented the fiber $\mathcal{F}_b$.

1.2.1. *How can studying* $\overline{\mathrm{M}}_g$ *tell us about curves?* Earlier we considered a family of curves parametrized by an open subset of $\mathbf{A}^6$, that included the general smooth curve of genus 2. Generally, if there is a family of curves parametrized by an open subset of $\mathbf{A}^{N+1}$ that includes the general curve of genus g, then one would have a dominant rational map from $\mathbf{P}^N$ to our compactification $\overline{\mathrm{M}}_g$. In other words, $\overline{\mathrm{M}}_g$ would be unirational. This would imply that there are no pluricanonical forms on $\overline{\mathrm{M}}_g$. Said otherwise still, the canonical divisor of $\overline{\mathrm{M}}_g$ would not be effective.

On the other hand, one of the most important results about the moduli space of curves, proved almost 40 years ago, is that for $g >> 0$ the canonical divisor of $\overline{\mathrm{M}}_g$ lives in the interior of the cone of effective divisors (for $g = 22$ and $g \geq 24$, by [EH87, HM82], and for by $g = 23$ [Far00]). Once the hard work was done to write down the classes of the canonical divisor, and an effective divisor called the Brill-Noether locus, to prove this famous result, a very easy combinatorial argument can be made to show that the canonical divisor is equal to an effective linear combination of the Brill-Noether and boundary divisors when the genus is large enough.

The upshot is that by shifting focus to the geometry of the moduli space of curves, we learn something basic and valuable about the existence of equations of smooth general curves. Nevertheless, basic open questions remain. First, our current understanding of such questions is incomplete – it can be summarized in the following picture:



So there is a gap in our understanding of the "nature" of $\overline{\mathrm{M}}_g$. On the other hand, even for those g for which we know the answer, there are still problems to solve. For instance if $\overline{\mathrm{M}}_g$ is known to be of general type, one can consider the canonical ring

$$\mathrm{R}_\bullet = \bigoplus_{m\geq 0}\Gamma(\overline{\mathrm{M}}_g, m\,\mathrm{K}_{\overline{\mathrm{M}}_g}),$$

which is now known to be finitely generated by the celebrated work of [BCHM10]. In particular, the canonical model $\mathrm{Proj}(\mathrm{R}_\bullet)$, is birational to $\overline{\mathrm{M}}_g$.

It is still an open problem to construct this model, and efforts to achieve this goal have both furthered our understanding of the birational geometry of the moduli space of curves, as well as giving a highly nontrivial example where this developing theory can be experimented with and better understood.

Remark 1.2.2. *We have described moduli spaces of curves as projective varieties. But in doing so we gloss over some of what makes them moduli spaces. There is a functorial way to describe moduli spaces which leads to their study as stacks.*

1.2.3. *A stratification.* As we have seen in the examples above, even if we are only interested in smooth curves, we are naturally led to curves with singularities, and when considering curves with nodes, one is naturally led to curves with *marked points*.

The moduli space $\overline{\mathrm{M}}_g$ is a (3g-3)-dimensional projective variety. The set $\delta^k(\overline{\mathrm{M}}_g) = \{[C] \in \overline{\mathrm{M}}_g |\ C$ has at least k nodes$\}$ has codimension k in $\overline{\mathrm{M}}_g$. If $k = 1$, these loci have codimension one, and the boundary is a union of components:

(1) The component Δ_{irr} can be described as having generic point with a nonseparating node; the closure of the locus of curves whose normalization is a curve of genus $g-1$ with 2 marked points.
(2) Components $\Delta_{g_1} = \Delta_{g_2}$ are determined by partitions $g = g_1 + g_2$. These loci can be described as having generic point with a separating node – the closure of the set of curves whose normalization consists of 1-pointed curves of genus g_1 (and g_2).

We may picture generic elements in these sets, and their normalizations, as follows:

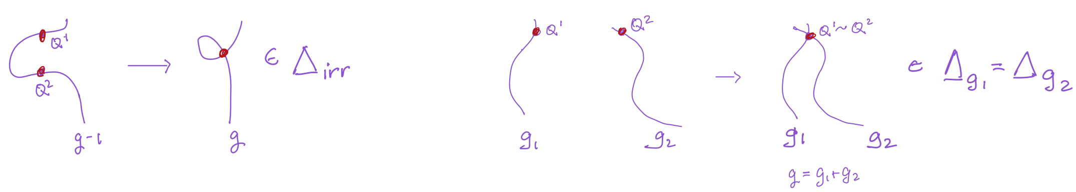


FIGURE 4. Clutching maps.

1.3. **Moduli of pointed curves.** By $\mathrm{M}_{g,n}$ we mean the quasi-projective variety whose points are in one-to-one correspondence with isomorphism classes of smooth n-pointed curves of genus $g \geq 0$. By the compactification $\overline{\mathrm{M}}_{g,n}$, we mean the variety whose points are in one-to-one correspondence with isomorphism classes of stable n-pointed curves of genus $g \geq 0$.

Definition 1.4. *A **stable n-pointed curve** is a complete connected curve C that has only nodes as singularities, together with an ordered collection p_1, p_2, ..., $p_n \in C$ of distinct smooth points of C, such that the $(n+1)$-tuple $(C; p_1, \ldots, p_n)$ has only a finite number of automorphisms.*

To get a sense of its combinatorial structure, we note that the moduli space is stratified by the topological type of the curves being parametrized. As we did last time in the case $n = 0$, we may describe these components of the boundary of $\overline{\mathrm{M}}_{g,n}$ as $\delta^k(\overline{\mathrm{M}}_{g,n}) = \{[(C, P^\bullet)] \in \overline{\mathrm{M}}_{g,n} |\ C$ has at least k nodes$\}$ in $\overline{\mathrm{M}}_{g,n}$ (a space of dimension $3g-3+n$). The locus $\delta^k(\overline{\mathrm{M}}_{g,n})$ has codimension k and is a union of irreducible components.

For instance, if $k = 1$, this codimension one locus is a union of components:

(1) Δ_{irr} has generic point a nonseparating node; the closure of the locus of curves whose normalization is a curve of genus $g-1$ with $n+2$ marked points.
(2) $\Delta_{g_1,N_1} = \Delta_{g_2,N_2}$ are determined by partitions $g = g_1 + g_2$ and $\{P^1, \ldots, P^n\} = N_1 \cup N_2$, with generic point a separating node – the closure of the set of curves whose normalization consists of pointed curves of genus g_1 (and g_2) with marked points in the set N_1 (and N_2)

As before, one can describe the components Δ_{irr} and Δ_{g_1,N_1} as the images of attaching maps from moduli spaces of stable curves with smaller genus (or with fewer marked points):

$$\overline{\mathrm{M}}_{g-1,n+2} \longrightarrow \Delta_{irr} \subset \overline{\mathrm{M}}_{g,n}, \quad \text{and} \quad \overline{\mathrm{M}}_{g_1,n_1+1} \times \overline{\mathrm{M}}_{g_2,n_2+1} \longrightarrow \Delta_{g_1,N_1} = \Delta_{g_2,N_2} \subset \overline{\mathrm{M}}_{g,n}$$

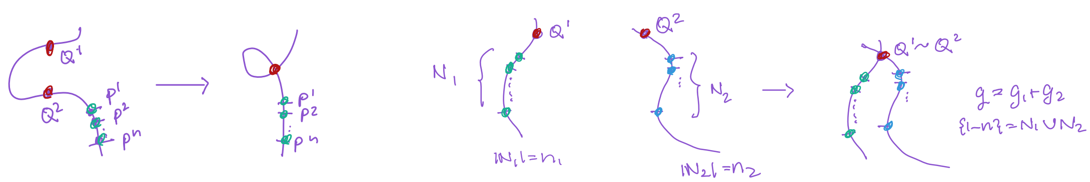


FIGURE 5. Tautological clutching maps.

There are also tautological point dropping maps, and using them we obtain $n+1$ families of stable n-pointed curves parametrized by $\overline{\mathrm{M}}_{g,n}$

$$\pi_j : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,n}, \quad s_i : \overline{\mathrm{M}}_{g,n} \to \overline{\mathrm{M}}_{g,n+1}, \quad i \in \{1, \ldots, n+1\} \setminus \{j\}$$

where π_j is the map that drops the j-th point, and s_i is the section that takes an n-pointed curve $(C; \vec{p})$ and at the i-th point attaches a copy of $\mathbb{P}^1$ labeled with two additional points p_i and p_{n+1}.

1.3.1. *Comparing* $\overline{\mathrm{M}}_{0,n}$ *with moduli spaces of higher genus curves.* The space $\overline{\mathrm{M}}_{0,n}$ has some advantages over $\overline{\mathrm{M}}_{g,n}$ for $g > 0$, for several reasons, three of which are easy to state. First $\overline{\mathrm{M}}_{0,n}$ is a fine moduli space (it parametrizes pointed curves with no nontrivial isomorphisms), unlike $\overline{\mathrm{M}}_{g,n}$ for $g > 0$, which parametrizes curves with non-trivial automorphism. Second, $\overline{\mathrm{M}}_{0,n}$ is smooth, whereas $\overline{\mathrm{M}}_{g,n}$ for $g > 0$ has singularities. So there are tools like intersection theory that are easier to carry out. Third, $\overline{\mathrm{M}}_{0,n}$ is rational (unlike $\overline{\mathrm{M}}_{g,n}$ for $g >> 0$), making some arguments easier.

There are a number of constructions of $\overline{\mathrm{M}}_{0,n}$, giving one different perspectives about the space, and tools to work with it. For instance, Kapranov showed $\overline{\mathrm{M}}_{0,n}$ is a Hilbert (or Chow quotient) of Veronese curves and can be seen as a quotient of a Grassmannian. There are at least four ways to construct the space as a sequence of blowups. Finn Knudsen was first to observe this, showing that $\overline{\mathrm{M}}_{0,n+2}$ could be constructed as a sequence of blowups of $\overline{\mathrm{M}}_{0,n+1} \times_{\overline{\mathrm{M}}_{0,n}} \overline{\mathrm{M}}_{0,n+1}$ (this product is not smooth), along non-regularly embedded subschemes. Keel improved this, giving an alternative construction of $\overline{\mathrm{M}}_{0,n}$ as a sequence of blowups of smooth varieties along smooth co-dimension 2 sub-varieties. The first case where we see anything interesting is for the 2-dimensional space $\overline{\mathrm{M}}_{0,5}$ which is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{P}^1 \times_{pt} \mathbb{P}^1$.

As a result of his construction, Keel in [Kee92] showed that Chow groups and homology groups are canonically isomorphic, giving recursive formulas for the Betti numbers, and an inductive recipe for the basis of Chow rings, which he shows are quotients of polynomial rings (he gives the generators and the relations). As an example, we know from Keel that there are $2^{n-1} - \binom{n}{2} - 1$ numerical (or linear, or algebraic) equivalence classes of codimension 1 classes (divisors) on $\overline{\mathrm{M}}_{0,n}$. One may also use the projection maps and facts about $\overline{\mathrm{M}}_{0,4}$, which is isomorphic to $\mathbb{P}^1$ to deduce numerical equivalences of divisors on $\overline{\mathrm{M}}_{0,n}$ for all n. For instance, since $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$, one has that on $\overline{\mathrm{M}}_{0,4}$, all boundary divisor classes are equivalent. So in particular,

$$\delta_{ij} \equiv \delta_{ik} \equiv \delta_{i\ell}, \text{ for } \{i,j,k,\ell\} = \{1,2,3,4\}.$$

One can show, using the point dropping maps, that for $n \geq 4$, on $\overline{\mathrm{M}}_{0,n}$,

$$\sum_{I \subset \{ijk\ell\}^c} \delta_{ij\cup I} \equiv \sum_{I \subset \{ijk\ell\}^c} \delta_{ik\cup I} \equiv \sum_{I \subset \{ijk\ell\}^c} \delta_{i\ell\cup I}, \text{ for any four indices } \{i,j,k,\ell\} \subset \{1,\dots,n\}.$$

1.4. **Moduli of stable coordinized curves.** $\widehat{\mathcal{M}}_{g,n}$ is the stack parametrizing families of tuples $(C, P_\bullet, t_\bullet)$, where $(C, P_\bullet = (P_1,\dots,P_n))$ is a stable n-pointed genus g curve, and $t_\bullet = (t_1,\dots,t_n)$ with t_i a formal coordinate at P_i, for each i. A description of $\widehat{\mathcal{M}}_{g,n}$ over the locus of smooth curves is given in [ADCKP88] and [FBZ04, §6.5], and over the locus parametrizing stable curves with singularities in [DGT21, §2]. By dropping the coordinates, there is a projection $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. A group scheme $(\mathrm{Aut}\,\mathcal{O})^{\oplus n}$, defined below, acts transitively on fibers by change of coordinates, giving $\widehat{\mathcal{M}}_{g,n}$ the structure of an $(\mathrm{Aut}\,\mathcal{O})^{\oplus n}$-torsor over $\overline{\mathcal{M}}_{g,n}$. Moreover, $\mathrm{Aut}\,\mathcal{O} = \mathbb{G}_m \ltimes \mathrm{Aut}_+\mathcal{O}$, and the projection factors as a composition of an $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$-torsor and a $\mathbb{G}_m^{\oplus n}$-torsor:

$$
\begin{tikzcd}
\widehat{\mathcal{M}}_{g,n} \arrow[rr, "(\mathrm{Aut}\,\mathcal{O})^{\oplus n}"] \arrow[dr, "(\mathrm{Aut}_+\mathcal{O})^{\oplus n}"'] & & \overline{\mathcal{M}}_{g,n} \\
& \overline{\mathcal{J}}_{g,n}. \arrow[ur, "\mathbb{G}_m^{\oplus n}"'] &
\end{tikzcd}
\tag{1}
$$

Here $\overline{\mathcal{J}}_{g,n}$ denotes the stack parametrizing families of pointed curves with first order tangent data. Closed points in $\overline{\mathcal{J}}_{g,n}$ are denoted $(C, P_\bullet, \tau_\bullet)$, where $(C, P_\bullet)$ is a stable n-pointed curve of genus g, and $\tau_\bullet = (\tau_1, \dots, \tau_n)$ with τ_i a non-zero 1-jet at a formal coordinate at P_i, for each i.

Remark 1.4.1. *The sheaf of coinvariants on $\overline{\mathcal{M}}_{g,n}$ is defined first on $\widehat{\mathcal{M}}_{g,n}$, shown to descend to $\overline{\mathcal{J}}_{g,n}$, and then if conditions are right, shown to descend to $\overline{\mathcal{M}}_{g,n}$ using Tsuchimoto's method, as described carefully* [DGT22a, §8]. *At the moment, the complete description of the descent is given only in case the conformal dimensions of modules are rational numbers. We are working on descent in greater generality, but for this reason, I will point out assumptions on V or categories of V-modules where this condition is known to hold.*

The group schemes discussed above represent functors. For instance, $\mathrm{Aut}\,\mathcal{O}$ represents the functor which assigns to a $\mathbb{C}$-algebra R the group

$$\mathrm{Aut}\,\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = a_1 z + a_2 z^2 + \cdots \mid a_i \in R,\ a_1 \text{ a unit} \right\}$$

of continuous automorphisms of the algebra $R[\![z]\!]$ preserving the ideal $zR[\![z]\!]$. The group law is given by composition of series: $\rho_1 \cdot \rho_2 := \rho_2 \circ \rho_1$. The subgroup scheme $\mathrm{Aut}_+\mathcal{O}$ of $\mathrm{Aut}\,\mathcal{O}$ represents the functor assigning to a $\mathbb{C}$-algebra R the group:

$$\mathrm{Aut}_+\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = z + a_2 z^2 + \cdots \mid a_i \in R \right\}.$$

To give more details about the actions, for a smooth curve C, let $\mathscr{A}ut_C$ be the smooth variety whose set of points are pairs (P, t), with $P \in C$, $t \in \widehat{\mathscr{O}}_P$ and $t \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$, a formal coordinate at P. Here $\mathfrak{m}_P$ is the maximal ideal of $\widehat{\mathscr{O}}_P$, the completed local ring at the point P. There is a simply transitive right action of $\mathrm{Aut}\,\mathcal{O}$ on $\mathscr{A}ut_C \to C$, given by changing coordinates:

$$\mathscr{A}ut_C \times \mathrm{Aut}\,\mathcal{O} \to \mathscr{A}ut_C, \qquad ((P,t),\rho) \mapsto (P, t\cdot\rho := \rho(t)),$$

making $\mathscr{A}ut_C$ a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle on C. A choice of formal coordinate at P gives a trivialization

$$\mathrm{Aut}\,\mathcal{O} \xrightarrow{\simeq_t} \mathscr{A}ut_P, \qquad \rho \mapsto \rho(t).$$

If C is a *nodal* curve, then to define a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle on C one may give a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle on its normalization, together with a gluing isomorphism between the fibers over the preimages of each node. For simplicity, suppose C has a single node Q, and let $\widetilde{C} \to C$ denote its normalization, with Q_+ and Q_- the two preimages of Q in $\widetilde{C}$. A choice of formal coordinates $s_\pm$ at $Q_\pm$, respectively, determines a smoothing of the nodal curve C over $\mathrm{Spec}(\mathbb{C}[\![q]\!])$ such that, locally around the point Q in C, the family is defined by $s_+s_- = q$. One may identify the fibers at $Q_\pm$ by the gluing isomorphism induced from the identification $s_+ = \gamma(s_-)$:

$$\mathscr{A}ut_{Q_+} \simeq_{s_+} \operatorname{Aut}\mathcal{O} \xrightarrow{\cong} \operatorname{Aut}\mathcal{O} \simeq_{s_-} \mathscr{A}ut_{Q_-}, \qquad \rho(s_+) \mapsto \rho \circ \gamma(s_-),$$

where $\gamma \in \operatorname{Aut}\mathcal{O}$ is the involution defined as

$$\gamma(z) := \frac{1}{1+z} - 1 = -z + z^2 - z^3 + \cdots .$$

This may be carried out in families, and the identification of the universal curve $\overline{\mathcal{C}}_g \cong \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ leads to the definition of the principal $(\operatorname{Aut}\mathcal{O})$-bundle $\widehat{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_{g,1}$ (see [DGT21] for details).

2. VOAs and their representations

I will describe VOAs and their modules, emphasizing properties relevant to the sheaves of conformal blocks defined from them.

2.1. A brief description of vertex operator algebras of conformal field theory type.

Definition 2.1. *A VOA of CFT-type is a four-tuple $V = (V, \mathbf{1}, \omega, Y(\cdot, z))$, where:*

(1) $V = \bigoplus_{i\in\mathbb{N}} V_i$ is a $\mathbb{N}$-graded $\mathbb{C}$–vector space with $\dim V_i < \infty$;
(2) $\mathbf{1}$ is an element in V_0, called the vacuum vector;
(3) ω is an element in V_2, called the conformal vector;
(4) $Y(\cdot, z)\colon V \to End(V)[\![z, z^{-1}]\!]$ is a linear map $A \mapsto Y(A, z) := \sum_{i\in\mathbb{Z}} A_{(i)} z^{-i-1}$. The series $Y(A, z)$ is called the vertex operator assigned to A.

The datum $\left(V, \mathbf{1}^V, \omega, Y(\cdot, z)\right)$ must satisfy the following axioms:

(a) (vertex operators are fields) for all $A, B \in V$, $A_{(i)}B = 0$, for $i \gg 0$;
(b) (vertex operators of the vacuum) $Y(\mathbf{1}^V, z) = id_V$:

$$\mathbf{1}^V_{(-1)} = id_V \qquad \text{and} \qquad \mathbf{1}^V_{(i)} = 0, \quad \text{for } i \neq -1,$$

and for all $A \in V$, $Y(A, z)\mathbf{1}^V \in A + zV[\![z]\!]$:

$$A_{(-1)}\mathbf{1}^V = A \qquad \text{and} \qquad A_{(i)}\mathbf{1}^V = 0, \qquad \text{for } i \geq 0;$$

(c) (weak commutativity) for all $A, B \in V$, there exists an $N \in \mathbb{Z}_{\geq 0}$ such that

$$(z - w)^N \left[Y(A, z), Y(B, w)\right] = 0 \quad \text{in } End(V)\left[\!\left[z^{\pm 1}, w^{\pm 1}\right]\!\right];$$

(d) (conformal structure) for $Y(\omega, z) = \sum_{i\in\mathbb{Z}} \omega_{(i)} z^{-i-1}$,

$$\left[\omega_{(p+1)}, \omega_{(q+1)}\right] = (p - q)\,\omega_{(p+q+1)} + \frac{c}{12}\,\delta_{p+q,0}\,(p^3 - p)\,id_V.$$

Here $c \in \mathbb{C}$ is the central charge of V. Moreover:

$$\omega_{(1)}|_{V_i} = i \cdot id_V, \quad \text{for all } i, \qquad \text{and} \qquad Y\left(\omega_{(0)}A, z\right) = \partial_z Y(A, z).$$

Various measures of finiteness that have been developed to distinguish VOAs. For instance

Definition 2.2. *A vertex algebra V is called finitely strongly generated if there exists finitely many elements $A^1, \ldots, A^r \in V$ such that V is spanned by the elements of the form*

$$A^{i_1}_{(-n_1)} \cdots A^{i_r}_{(-n_r)}\mathbf{1},$$

with $r \geq 0$ and $n_i \geq 1$ (see [Ara12]*). We say that $V = \bigoplus_{i\in\mathbb{N}} V_i$ is strongly generated in degree d if it is possible to choose the generators A^{i_j} to be in V_m for $m \leq d$.*

As it turns out if V is C_1-cofinite (this is another measure of finiteness, defined soon), then V is (strongly) finitely generated. After giving a couple of examples, I will define V-modules, and then quotients of V that are easier to work with and and encode important information about the V-modules.

2.2. **Examples.** While their constructions are analogous, affine VOAs are *strongly generated in degree 1*, while Virasoro VOAs have no degree 1 component, and are generated in degree 2 by their Virasoro vector. Affine VOAs behave differently depending on the Lie algebra $\mathfrak{g}$ and the level ℓ used to define them. For instance, if $\mathfrak{g}$ is a reductive Lie algebra, and $\ell \in \mathbb{Z}_{>0}$, then affine VOAs are rational and C_2-cofinite (aka *lisse*). A VOA is rational if every admissible module is completely reducible. That V is C_2-cofinite implies the central charge of V and conformal dimensions of V-modules are rational (shown in a physical context by Moore and Anderson [AM88] (and this may be the origin of the confusion of the word rationality). If k is an admissible and not integral, then the VOAs are not lisse, but they are *quasi-lisse*. Moreover, modules from the very nice category $\mathcal{O}$ have rational conformal dimension and complete reducibility holds there. We next briefly define these two types of examples.

2.2.1. *Virasoro VOA.* To construct the Virasoro VOA, we take the affinization of the Virasoro Lie algebra. Let $\mathrm{Vir}_{\geq 0} := \mathbb{C}K \oplus z\mathbb{C}[\![z]\!]\partial_z$ be a Lie subalgebra of the Virasoro Lie algebra Vir, and let $M_{c,h} := U(\mathrm{Vir}) \otimes_{U\left(\mathrm{Vir}_{\geq 0}\right)} \mathbb{C}\mathbf{1}$ be the Verma module of highest weight $h \in \mathbb{C}$ and central charge $c \in \mathbb{C}$ ($M_{c,h}$ is a module over $M_{c,0}$). There is a unique maximal proper submodule $J_{c,h} \subset M_{c,h}$. Set $L_{c,h} := M_{c,h}/J_{c,h}$, and $\mathit{Vir}_c := L_{c,0}$. By [Wan93, Theorem 4.2 and Corollary 4.1], one has that Vir_c is rational if and only if $c = c_{p,q} = 1-6\frac{(p-q)^2}{pq}$, where p and $q \in \{2, 3, \ldots\}$ are relatively prime. By [DLM00, Lemma 12.3] (see also [Ara12, Proposition 3.4.1]) Vir_c is C_2-cofinite for $c = c_{p,q}$, and by [FZ92, Theorem 4.3], Vir_c is of CFT-type. By [Wan93, Theorem 4.2] the modules $L_{c,h}$ are irreducible if and only if

$$h = \frac{(np-mq)^2-(p-q)^2}{4pq}, \text{ with } 0 < m < p, \text{ and } 0 < n < q.$$

Note that by definition h is the conformal dimension of $L_{c,h}$. These are unitary if $|q-p| = 1$.

2.2.2. *Affine VOAs.* Here we briefly discuss affine VOAs V, which together with their V-modules, define the most well-studied examples of sheaves of coinvariants (and dual sheaves of conformal blocks). For complete details, see [TUY89], [FZ92], and [Lia94]. VOAs of CFT-type, strongly generated in degree 1 were classified in [Lia94]. For any so-called preVOA V of CFT-type by [Lia94, Theorem 3.7], the degree 1 component V_1 has the structure of a Lie algebra, with bracket $[A, B] = A_{(0)}(B)$. Moreover, this Lie algebra $(V_1, [\ ,\ ])$, which need not be simple, or reductive, is equipped with a symmetric invariant bilinear form $< A\ ,\ B >= A_{(1)}(B)$. Roughly speaking, in the terminology of [Lia94], a preVOA satisfies many of the properties of a VOA except those involving a conformal vector. Given any pair consisting of a Lie algebra $\mathfrak{g}$ and symmetric invariant bilinear form $<\ ,\ >$, Following [FZ92], Lian defines the affinization, and proves in [Lia94, Theorem 4.11] that for any preVOA V of CFT-type, if strongly generated in degree 1, then V is isomorphic to a quotient of the affinization of $(V_1, <\ ,\ >)$ by some ideal. The last step in the classification is to determine which preVOAs admit a Virasoro vector, and have the structure of a VOA. He classifies such Virasoro vectors (see [Lia94, Corollary 6.15]). Arguably the most interesting aspect of [Lia94] is that this class is much richer than the affinizations of reductive Lie algebras and their quotients. New examples showing that this class is much richer than universal and simple affine VOAs constructed from reductive Lie algebras as in [FZ92]. New examples are given in [Lia94, §6.4]. We next briefly describe the affine Lie algebra $\mathfrak{g}$ via the (normalized) Killing form. As demonstrated in [Lia94], analogous affinizations can be defined for any Lie algebra $(V_1, [\ ,\ ]_{V_1})$ on which there is a symmetric invariant bilinear form $<\ ,\ >$.

Given a simple Lie algebra $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ over $\mathbb{C}$, and the Cartan-Killing form[1] $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, which conventionally, one normalizes so that $\frac{(\theta,\theta)}{2} = 1$, and a formal parameter c, we consider

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus\ \mathbb{C} \cdot c,$$

a one-dimensional central extension of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}((\xi))$, where $\mathbb{C}((\xi))$ is the field of Laurent series. Elements in $\widehat{\mathfrak{g}}$ are tuples $(a, \alpha c)$, with $a = \sum_j \mathrm{X}_j \otimes f_j$, with $f_j \in \mathbb{C}((\xi))$. We define the bracket on simple tensors so that $[(\mathrm{X} \otimes f, \alpha c), (0, c)] = 0$ and c is in the center of $\hat{\mathfrak{g}}$, and

$$[(\mathrm{X} \otimes f, \alpha c), (\mathrm{Y} \otimes g, \beta c)] = \Bigg([\mathrm{X}, \mathrm{Y}] \otimes fg\ ,\ \underset{\text{Killing}}{c(\mathrm{X}, \mathrm{Y})} \cdot \mathrm{Res}_{\xi_i=0}(g(\xi) df(\xi)) \Bigg).$$

The affine Lie algebra $\widehat{g}$ has a triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_{<0} \oplus (\mathfrak{g} \otimes \mathbb{C}c) \oplus \widehat{\mathfrak{g}}_{>0}$, where

$$\widehat{\mathfrak{g}}_{<0} = \mathfrak{g} \otimes \xi^{-1}\mathbb{C}[\xi^{-1}],\ \widehat{\mathfrak{g}}_{>0} = \mathfrak{g} \otimes \xi\mathbb{C}[[\xi]],\ \text{and}\ \widehat{\mathfrak{g}}_{\geq 0} = \mathfrak{g} \otimes \mathbb{C}c \oplus \mathfrak{g} \otimes \xi\mathbb{C}[[\xi]],$$

are Lie subalgebras. We note that $\widehat{\mathfrak{g}}_{<0}$ is generated by elements of the form $X \otimes \xi^{-m}$, where $X \in \mathfrak{g}$, and for $m \in \mathbb{N}_{\geq 0}$ and by convention, such an element $X \otimes \xi^{-m} := X_{(m)}$ has degree $\deg(X) + m - 1 = m \geq 0$, since all elements $X \in \mathfrak{g}$ are taken to have degree 1.

Following [TUY89], to every $\mathfrak{g}$-module[2] V^λ, one can form the Verma module, a $\mathcal{U}(\widehat{\mathfrak{g}})$-module[3]

$$\mathcal{M}^\lambda := \mathcal{U}(\widehat{\mathfrak{g}}) \otimes_{\mathcal{U}((\widehat{\mathfrak{g}})_{\geq 0})} V^\lambda \underset{\text{PBW}}{\cong} U((\widehat{\mathfrak{g}})_{<0}) \otimes_{\mathbb{C}} V^\lambda.$$

For this, one extends the action of $\mathfrak{g}$ on V^λ to an action of $\widehat{\mathfrak{g}}_{\geq 0}$ on V^λ by declaring that $\widehat{g}_{>0}$ act by zero, and the central element c by $\ell \cdot id_{V^\lambda}$. In particular, from the trivial $\mathfrak{g}$-module with generator $1 \in \mathfrak{g}$, one obtains the universal affine VOA $V = V_\ell(\mathfrak{g}) \cong U(\widehat{\mathfrak{g}}) \otimes_{\mathcal{U}((\widehat{\mathfrak{g}})_{\geq 0})} 1 \cong \mathcal{U}((\widehat{\mathfrak{g}})_{<0}) \otimes_{\mathbb{C}} 1$, whose elements are (linear combinations of) strings $X^1_{(m_1)} \cdot X^2_{(m_2)} \cdots X^i_{(m_i)} \cdot 1$, of degree

$$\deg(X^1_{(m_1)} \cdot X^2_{(m_2)} \cdots X^i_{(m_i)} \cdot 1) = \sum_{j=1}^{i} m_i \geq 0.$$

The vacuum element is the degree 0 element $\mathbf{1}^V = 1$ where X is the identity element of $\mathfrak{g}$. The Virasoro vector is given by the so-called Sugawara construction. For a basis $\mathcal{B}$ of $\mathfrak{g}$, orthonormal with respect to the (normalized) Killing form, the conformal vector is defined to be

$$\omega = \frac{1}{2(h^\vee + \ell)} \sum_{B \in \mathcal{B}} : B_{(-1)} B_{(-1)} := \frac{1}{2(h^\vee + \ell)} \sum_{B \in \mathcal{B}} B^2_{(-1)}.$$

[1]This symmetric invariant bilinear form has the property that $([X, Y]_{\mathfrak{g}}, Z) + (Y, [X, Z]_{\mathfrak{g}}) = 0$.

[2]There is a 1-1 correspondence between finite dimensional irreducible representations of $\mathfrak{g}$ and dominant integral weights of $\mathfrak{g}$. These define a full-subcategory of irreducible representations of the affine Lie algebra $\widehat{\mathfrak{g}}$.

[3]$\mathcal{U}(\mathfrak{L})$ is the completion of the universal enveloping algebra associated to a Lie algebra $\mathfrak{L}$ as defined in [FZ92].

and vertex operators defined for elements $A = \sum_j X^{1,j}_{(m_{1j})} \cdot X^{2,j}_{(m_{2,j})} \cdots X^{i,j}_{(m_{i,j})} \cdot 1$

$$Y(A,z) = \sum_{m\in\mathbb{Z}} A_{(m)} z^{-m-1},$$

where the endomorphism $A_{(m)}$ acts by cancatination. Give an example.

Generators $\{L_p\}_{p\in\mathbb{Z}}$ of the associated Virasoro Lie-algebra Vir acting on V, are defined in this case by $L_p := \frac{1}{2(h^\vee+\ell)} \sum_{m\in\mathbb{Z}} \sum_{B\in\mathcal{B}} : B_{(m)} B_{(p-m)} :$ such that $:\ :$ denotes the normal ordering

$$:X_{(n)}Y_{(m)}: = \begin{cases} X_{(n)}Y_{(m)} & \text{if } n<m \\ \frac{1}{2}\left(X_{(n)}Y_{(m)} + Y_{(m)}X_{(n)}\right) & \text{if } n=m \\ Y_{(m)}X_{(n)} \text{if } n>m. \end{cases} \tag{2}$$

By [TUY89, Lemma 1.2.2], the set $\{L_p\}$ generates a Virasoro Lie algebra such that

$$[L_p, L_q] = (p-q)L_{p+q} + \frac{c_V}{12}(p^3-p)\delta_{p+q,0}, \quad \text{with} \quad c_V = \frac{\ell \dim(\mathfrak{g})}{h^\vee+\ell}.$$

Recall that c_V is the central charge of V. The universal affine VOA $V_\ell(\mathfrak{g})$ has a unique maximal submodule Z, and after taking the quotient, one obtains the simple affine VOA $L_\ell(\mathfrak{g}) = V_\ell(\mathfrak{g})/Z$.

2.3. **Modules.** There are a number of ways to define V-modules. Following [NT05], V-modules are certain modules over $\mathscr{U}(V)$, the (completed) universal enveloping algebra defined in [FZ92, §1.3], called the current algebra in [NT05, §2.2]. This is an associative algebra, topologically generated by the enveloping algebra a Lie algebra $\mathfrak{L}(V)$, described next.

As a vector space, $\mathfrak{L}(V)$ is the quotient $(V \otimes \mathbb{C}(\!(t)\!))/ \operatorname{im} \nabla$, where

$$\nabla : V \otimes \mathbb{C}(\!(t)\!) \to V \otimes \mathbb{C}(\!(t)\!), \qquad v \otimes f \mapsto L_{-1}v \otimes f + v \otimes \frac{df}{dt}.$$

Here $L_{-1} = \omega_{(0)}$ is the coefficent of z^{-1} in the power series $Y(\omega,z) = \sum_{m\in\mathbb{Z}} \omega_{(m)} z^{-m-1}$. This operator is like a derivative, since from the axioms for a VOA, for any $v \in V$, one has $Y(L_{-1}v,z) = \frac{d}{dz}Y(v,z)$. The bracket for the Lie algebra $\mathfrak{L}(V)$ is defined on generators $v_{[i]} = \overline{v \otimes t^i}$, and $u_{[j]} = \overline{u \otimes t^j}$ by Borcherd's identity, also a consequence of the axioms:

$$[v_{[i]}, u_{[j]}] = \sum_{k=0}^{\infty} \binom{i}{k} (v_{(k)}(u))_{[i+j-k]}.$$

There is a triangular decomposition $\mathfrak{L}(V) = \mathfrak{L}(V)_{<0} \oplus \mathfrak{L}(V)_0 \oplus \mathfrak{L}(V)_{>0}$, where

$$\mathfrak{L}(V)_{<0} = \text{Span}\left\{v_{[i]} \in \mathfrak{L}(V) \ : \ \deg(v_{[i]}) = \deg(v) - i - 1 < 0\right\},$$
$$\mathfrak{L}(V)_{>0} = \text{Span}\left\{v_{[i]} \in \mathfrak{L}(V) \ : \ \deg(v_{[i]}) = \deg(v) - i - 1 > 0\right\}, \text{ and}$$
$$\mathfrak{L}(V)_0 = \text{Span}\left\{v_{[\deg(v)-1]} \ : \ v \in V \text{ homogeneous}\right\}.$$

By [NT05, Definition 2.3.1] a V-module W is a finitely generated $\mathscr{U}(V)$-module such that for any $w \in W$, the vector space $F^0\mathscr{U}(V)w$ is a finite-dimensional vector space, and there is a positive integer d such that $F^d\mathscr{U}(V)w = 0$, where the filtration is induced from that of $\mathfrak{L}(V)$ (see eg [NT05, §2.2]).

As is explained in [NT05], the filtration of $\mathscr{U}(V)$ allows one to show that V-modules are $\mathbb{N}$-gradable. These are called admissible modules and grading restricted weak V-modules in the literature. We will refer to them as *V-modules*.

Such modules can also be described as pairs $\big(W, Y^W(-,z)\big)$ consisting of

(1) an $\mathbb{N}$-graded vector space $W = \bigoplus_{i\geq 0} W_i$ with finite dimensional graded pieces W_i and $W_0 \neq 0$;
(2) a linear map $Y^W(-,z)\colon V \to \mathrm{End}(W)[\![z,z^{-1}]\!]$ which sends an element $A \in V$ to $Y^W(A,z) = \sum_{i\in\mathbb{Z}} A^W_{(i)} z^{-i-1}$.

In order for this pair to define an admissible V-module, certain axioms need to hold [DGT22a, FHL93, DL93]. Instead of reporting all the properties that $(W, Y^W(z,-))$ must satisfy, we list here only those some we refer to [DGT22a] for more details.

(1) *conformal structure:* the Virasoro algebra acts on W through the identification $L_p \cong \omega^W_{(p+1)}$.
(2) *vacuum axiom:* $Y^W(\mathbf{1}, z) = \mathrm{Id}_W$.
(3) *graded action:* if $A \in V_k$, then $A^W_{(j)} W_\ell \subseteq W_{\ell+k-j-1}$ and we write $\deg(A^W_{(j)}) = \deg(A) - j - 1$.
(4) *commutator formula:* $[A^W_{(i)}, B^W_{(j)}] = \sum_{k\geq 0} \binom{i}{k} \big(A_{(k)}(B)\big)^W_{(i+j-k)}$.
(5) *associator formula:* $\big(A_{(i)}(B)\big)^W_{(j)} = \sum_{k\geq 0} (-1)^k \binom{i}{k} \Big(A^W_{(i-k)} B^W_{(j+k)} - (-1)^i B^W_{(i+j-k)} A^W_{(k)}\Big)$.

In what follows the endomorphism $A^W_{(j)}$ will simply be denoted by $A_{(j)}$. It is important to observe that V is a V-module and that the commutator and associator formulas for V and for V-modules both arise from the *Jacobi identities* for V and for V-modules. Moreover, when W is a simple V-module, then there exists $\alpha \in \mathbb{C}$ called the *conformal dimension* such that $L_0(w) = (\alpha + \deg(w))w$ for every homogeneous element $w \in W$.

Definition 2.3. *An n-tuple $(W^1, \dots, W^n)$ of admissible V-modules W^i of conformal dimension α_i is said to satisfy the integrality condition, or integrality property, if $\sum_{i=1}^n \alpha_i$ is an integer (which can be zero).*

2.4. **Zhu's associative algebra and higher degree analogues.** There is an associative algebra $A(V)$, and a functor that takes $A(V)$-modules to V-modules, such that indecomposable $A(V)$-modules are taken to indecomposable V-modules, and giving a 1-1 correspondence between simple $A(V)$-modules and simple V-modules. We define these here.

We set, for $n \geq 0$, and for $a, b \in V$

$$a \circ_n b = \sum_{i=0}^{\infty} \binom{\deg(a)+n}{i} a_{i-2n-2}(b),$$

and let $A_n(V)$ be the quotient $V/O_n(V)$, where

$$O_0(V) = \mathrm{span}_{\mathbb{C}}\{a \circ_0 b \,|a, b \in V\}, \ \text{ if } n \geq 1,\ O_n(V) = \mathrm{span}_{\mathbb{C}}\{a \circ_n b, L_{(-1)}a + L_{(0)}b \,|a, b \in V\}.$$

For
$$a \star_n b = \sum_{m=0}^{n}\sum_{i=0}^{\infty}(-1)^m \binom{m+n}{n}\binom{\deg(a)+n}{i} a_{i-m-n-1}(b),$$
the pair $(A_n(V), \star_n)$ is an associative algebra generalizing what is frequently referred to as Zhu's algebra $A(V) = A_0(V)$, studied in [FZ92]. For $n \geq 1$, these were defined in [DLM97], and some statements later clarified and corrected in [BVWY19].

As shown in [DLM97], for each pair of indices $d \geq e$, since $O_e(V) \subset O_d(V)$, there are projections $\alpha_{d,e} : A_d(V) \twoheadrightarrow A_e(V)$, and these satisfy $\alpha_{d,d} = id_{A_d(V)}$, and $\alpha_{e,f} \circ \alpha_{d,e} = \alpha_{d,f}$ for $d \geq e \geq f$, and so these algebras form an inverse system
$$\cdots \twoheadrightarrow A_{d+1}(V) \twoheadrightarrow A_d(V) \twoheadrightarrow \cdots \twoheadrightarrow A_1(V) \twoheadrightarrow A_0(V) \to 0.$$
So there is an inverse limit $A_\infty(V)$ together with homomorphisms $\chi_d : A_\infty(V) \to A_d(V)$, such that $\alpha_{d,e} \circ \chi_d = \chi_e$, whenever $d \geq e$, and satisfying the following universal property. Given any associative algebra B and homomorphism $\beta_d : B \to A_d(V)$, satisfying $\alpha_{d,e} \circ \beta_d = \beta_e$ for $d \geq e$, there is a unique algebra homomorphism $\beta : B \to A_\infty(V)$, making the natural associated diagram commute.

Suppose that $\{M(d)\}_{d\in\mathbb{N}}$ is an inverse system of abelian groups such that each $M(d)$ is an $A_d(V)$ module, and such that for $d \geq e$ there are maps $\pi_{d,e} : M(d) \to M(e)$ that satisfy
$$\pi_{d,e}(a \cdot m) = \alpha_{d,e}\pi_{d,e}(m),$$
for $a \in A_d(V)$ and $m \in M(d)$. Then the inverse limit $M(\infty)$ of $M(d)$ is naturally an $A_\infty(V)$-module, and the action is continuous.

Given an admissible V-module $W = \sum_{d\in\mathbb{N}} W_d$, we obtain such an inverse system:
$$\Pi_d(W) = \oplus_{e=0}^{d} W_e, \text{ and for } d \geq e, \text{ and projection maps } \pi_{d,e} : \Pi_d(W) \to \Pi_e(W).$$
By [BVWY19, Theorem 3.1], for a nonzero $A_d(V)$-module E satisfying the property that for $d > 0$, E does not factor through $A_{d-1}(V)$, then $\mathcal{M}_d(E)$ is an $\mathbb{N}$-gradable V-module with $(\mathcal{M}_d(E))_0 \neq 0$. If there is no nonzero submodule of E that factors through $A_{d-1}(V)$, then $\Omega_d/\Omega_{d-1}(\mathcal{M}_d(E)) \cong E$. In Prop 3.9 they show that if E is an indecomposable $A_d(V)$-module that does not factor through $A_{d-1}(V)$, then $\mathcal{M}_d(E)$ is an indecomposable $\mathbb{N}$-gradable V-module generated by its degree d subspace (with simple modules corresponding to simple modules).

For this we need higher d generalizations of the Lie Algebra from [DGT22a, Eq (34)], and a Lemma about it shown for $d = 0$ in Step 2 of the proof of [DGT22a, Theorem 7.0.1].

2.5. **Functors.** We describe here Zhu's functor taking $A(V)$-modules to V-modules, and following [DGK22] a (compatible) functor taking $A(V)$-bimodules to $\mathscr{U}(V)^2$-modules. To do so, following [NT05, Theorem 3.3.5, (4) and (5)], let $\mathscr{U}(V)$ be the completion of the universal enveloping algebra for the Lie algebra $\mathfrak{L}(V)$, and recall the triangular decomposition of $\mathfrak{L}(V)$ (see §2.3). We denote by $\mathscr{U}(V)_{\leq 0}$ the sub-$\mathscr{U}(V)$ algebra, topologically generated by $\mathfrak{L}(V)_{<0} \oplus \mathfrak{L}(V)_0$. By [DLM98, Proposition 3.1], the map $\mathscr{U}(V)_0 \twoheadrightarrow A(V)$ given by $v_{[\deg v-1]} \mapsto v$ is surjective.

2.5.1. *Zhu's Functor.* Any $A(V)$-module E is a $\mathscr{U}(V)_0$-module, and the action of $\mathscr{U}(V)_0$ on E can be extended to an action of $\mathscr{U}(V)_{\leq 0}$ by letting $\mathfrak{L}(V)_{<0}$ act trivially. One then sets

$$\mathcal{M}(E) := \mathscr{U}(V) \otimes_{U(V)_{\leq 0}} E. \tag{3}$$

If E is simple, then $\mathcal{M}(E)$ has a unique, possibly zero, maximal sub-module $\mathcal{J}(E)$, and

$$L(E) := \mathcal{M}(E)/\mathcal{J}(E)$$

is simple, realizing the bijection between simple V-modules and simple $A(V)$-modules.

2.5.2. *Bimodule Functor.* Following [DGK22], we define $\Phi\colon A(V)$-bimod $\to \mathscr{U}(V)^{\otimes 2}$-mod as the functor which associates to every $A(V)$-bimodule E the $\mathscr{U}(V)^{\otimes 2}$-module

$$\Phi(E) = \mathrm{Ind}_{\mathscr{U}(\mathrm{V})_{\leq 0}^{\otimes 2}}^{\mathscr{U}(\mathrm{V})^{\otimes 2}} \mathrm{E} \cong \mathscr{U}(\mathrm{V})^{\otimes 2} \otimes_{\mathscr{U}(\mathrm{V})_{\leq 0}^{\otimes 2}} \mathrm{E}, \tag{4}$$

where the action of $a \otimes b \in \mathscr{U}(V)_{\leq 0}^{\otimes 2}$ on $e \in E$ is given by

$$(a \otimes b) \otimes e \mapsto \begin{cases} a \cdot e \cdot \theta(b) & \text{if } a, b \in \mathscr{U}(V)_0 \\ 0 & \text{if } a, b \in \mathscr{U}(V)_{<0}. \end{cases} \tag{5}$$

For b,$b' \in \mathscr{U}(V)_0$, $\theta(bb') = \theta(b')\theta(b)$, giving a bimodule action [NT05, Proposition 4.1.1].

Example 2.4. *The triplet algebras $\mathcal{W}(p)$ form an important family of non-rational, C_2-cofinite VOAs, which are strongly finitely generated in degree $2p-1$ by 1, ω, and three elements in $2p-1$, for $p \in \mathbb{Z}_{\geq 2}$. There are $2p$ non-isomorphic simple $\mathcal{W}(p)$-modules called by different notation in the literature including $\{\Lambda(i), \Pi(i)\}_{i=1}^{p}$ in* [AM88] *(and $\{X_s^{\pm} : 1 \leq s \leq p\}$ in eg.* [NT11, TW13]*). The corresponding simple $A(\mathcal{W}(p))$-modules are denoted $\{\Lambda(i)_0, \Pi(i)_0\}_{i=1}^{p}$ in* [AM88] *(and by $\{\overline{X}_s^{\pm} : 1 \leq s \leq p\}$ in* [NT11, TW13]*). By* [AM88, NT11]*, there is a bimodule decomposition*

$$A(\mathcal{W}(p)) \cong \bigoplus_{i=1}^{2p} B_i, \tag{6}$$

where components B_i are described as follows:

- *For $1 \leq i \leq p-1$, we have that $B_i \cong \mathbb{C}[\epsilon]/\epsilon^2 \cong \mathbb{I}_{h_i,1}$, which is indecomposable and reducible. In* [NT11, TW13] *this is denoted by $\widetilde{\overline{X}}_i^{+}$ and it is the projective cover of the simple $A(\mathcal{W}(p))$-module $\overline{X}_i^{+}$.*
- *$B_p \cong \mathbb{C} \cong \Lambda(p)_0 \otimes \Lambda(p)_0^{\vee}$.*
- *For $1 \leq i \leq p$, we have that $B_{p+i} \cong M_2(\mathbb{C}) \cong \Pi(i)_0 \otimes \Pi(i)_0^{\vee}$.*

The generalized Verma modules induced from the irreducible indecomposable $A(\mathcal{W}(p))$-modules $\Lambda(p)_0$, and $\Pi(p)_0$, are simple (see [AM88, page 2678]*). In particular, $\Lambda(p) = \mathcal{M}(\Lambda(p)_0) = L(\Lambda(p)_0)$ and $\Pi(p) = L(\mathcal{M}(\Pi(p)_0) = (\Pi(p)_0))$* [AM09].

2.6. **Standard finiteness conditions.** V is

(1) C_2-*cofinite* if and only if $\dim(V/C_2(V)) < \infty$, where $C_2(V) := \mathrm{span}_{\mathbb{C}} \left\{ v_{(-2)}u \, : \, v, u \in V \right\}$.
V C_2-cofinite, $\implies \dim(A(V)) < \infty$ [GN03].

(2) C_1-*cofinite* if and only if $\dim(V_+/C_1(V)) < \infty$, where

$$V_+ = \bigoplus_{d \in \mathbb{N}_{\geq 1}} V_d, \text{ and } C_1(V) = \mathrm{Span}_{\mathbb{C}}\{v_{(-1)}(u), L_{(-1)}(w) \mid v, u \in V_+, w \in V\}.$$

V is C_2-cofinite $\implies$ V is C_1-cofinite, and by [KL99, Proposition 3.2], if V is C_1-cofinite $\implies$ V is strongly finitely generated, and $A(V)$ is finitely generated.

(3) *rational* if and only if any V-module is a finite direct sum of simple V-modules.
V is rational $\implies$ $A(V)$ is finite and semi-simple, by [Zhu96, Theorem 2.2.3].

(4) *strongly rational* if V is simple, self-dual, C_2-cofinite, and rational.

(5) *strongly finite* if V is simple, self-dual, and C_2-cofinite.

3. SHEAVES OF COINVARIANTS AND DUAL SHEAVES OF CONFORMAL BLOCKS

When studying an object X in any category, it is important to determine the maps admitted by it. One approach is to find vector bundles on X, whose sections can be interpreted as (twisted) functions. Since the work of Tsuchiya, Ueno, and Yamada, and subsequent findings by a number of researchers, we know that representations of VOAs satisfying certain conditions define vector bundles on the moduli space of curves.

In these notes I will outline our contribution to this story, describing results from [DGT21, DGT22a, DGT22b, DG21] joint work with Damiolini, Tarasca, and recently [DGK22] with Damiolini and Krashen resulting in a reinterpretation of our original results. Vertex operator algebras (VOAs) are ubiquitous, important in many areas of mathematics and mathematical physics. VOAs and their representations are described in Lecture 2. Sheaves of coinvariants and their dual sheaves of conformal blocks are initially defined on the stack $\widehat{\mathcal{M}}_{g,n}$ (described in §1.4), as a quotient of a constant bundle $W^\bullet \otimes \mathscr{O}_{\widehat{\mathcal{M}}_{g,n}}$ by the action of a sheaf of (Chiral) Lie algebras $\mathcal{L}_{\widehat{\mathcal{C}}_{g,n}}$. This quotient sheaf is then shown to descend, first to $\overline{\mathcal{J}}_{g,n}$, the stack (introduced in Lecture 1) parametrizing pointed curves with first order tangent data (the $\mathcal{J}$ is meant to stand for jets), then to $\overline{\mathcal{M}}_{g,n}$. There are various ways to ensure the sheaf descends. For instance, if the conformal weights of the modules are rational, the sheaves descend. In §3.1 I will describe the fibers of these sheaves, and in §3.2 discuss their (known) properties.

3.1. **Part 1: fibers.** Because we are working with the stack $\widehat{\mathcal{M}}_{g,n}$, it is sufficient to define the sheaves on a family of stable pointed coordinatized curves:

$$(C, P_\bullet, \tau_\bullet) = (C \to S, \{P_i : S \to C\}_{i=1}^n, \{\tau_i : S \to C\}_{i=1}^n).$$

Given such a family, the fiber $\mathbb{V}(V, W^\bullet)_{(C,P_\bullet,\tau_\bullet)}$ depends on: (1) the collection of V-modules W^i, defined in §2.3; and (2) the action of the Chiral Lie algebra $\mathcal{L}_{C\setminus P_\bullet}(V)$ on $\otimes W^i$, defined here.

3.1.1. *Lie algebras that act.* The action of the Chiral Lie algebra on $\otimes W^i$ is defined via the diagonal action of the sheaf of ancillary Lie algebras $\mathfrak{L}(V)^n$ on the constant sheaf $(\otimes W^i) \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{g,n}}$. The fiber of the sheaf of ancillary Lie algebras at $(C, P_\bullet, \tau_\bullet)$ is the direct sum of Lie algebras $\mathfrak{L}_{P_i}(V)$, defined in §2.3. By [DGT22a] there is a coordinate independent version of $\mathfrak{L}_{P_i}(V)$, which gives a useful perspective for the definition of the Chiral Lie algebra, and which we briefly recount.

If t_i is a local coordinate at P_i, then the ancillary Lie algebra of V at P_i is isomorphic to

$$\mathfrak{L}_{P_i}(V) \cong \mathrm{H}^0\left(D_{P_i}^\times, \mathcal{V}_C \otimes \omega_C / \mathrm{Im}\,\nabla\right).$$

The sheaf $\mathcal{V}_C$ was originally defined on smooth curves in [FBZ04]. A definition of the sheaf $\mathcal{V}_C$ at a nodal curve is given in [DGT22a] (an alternative is given in [DGT21]). The construction from [DGT22a] yields a useful description of elements of $\mathcal{L}_{C\setminus P_\bullet}(V)$ in terms of the normalization of the curve. For simplicity, assume that C has a single node Q and let Q_+ and Q_- be the points lying above Q in the normalization $\widetilde{C}$. Then $\mathcal{L}_{C\setminus P_\bullet}(V)$ is the subquotient of

$$\bigoplus_{k\in\mathbb{N}} V_k \otimes \mathrm{H}^0\left(\widetilde{C}\setminus P_\bullet, \Omega_{\widetilde{C}}^{1-k}\otimes \mathscr{O}_C(-(k-1)(Q_+ + Q_-))\right). \tag{7}$$

of elements σ for which $[\sigma_{Q_+}]_0 = -\theta[\sigma_{Q_-}]_0$, where θ is an involution defined in [DGT21,DGT22a].

The chiral Lie algebra for $(C, P_\bullet)$ is then defined to be

$$\mathcal{L}_{C\setminus P_\bullet}(V) := \mathrm{H}^0\left(C\setminus P_\bullet, \mathcal{V}_C\otimes\omega_C/\mathrm{Im}\nabla\right).$$

One has a map $\mathcal{L}_{C\setminus P_\bullet}(V)\to\bigoplus_{i=1}^n \mathfrak{L}_{P_i}(V)$ given by these isomorphisms and restriction

$$\mathrm{H}^0(C\setminus P_\bullet, \mathcal{V}_C\otimes\Omega_C/\nabla_C)\longrightarrow\bigoplus_{i=1}^n \mathrm{H}^0(D_{P_i}^\times, \mathcal{V}_C\otimes\Omega_C/\nabla_C),\ \ \sigma\mapsto(\sigma_{P_i})_{i=1}^n.$$

To define the action of an image of such an element, recall that $\mathfrak{L}_{P_i}(V)$ is generated by images of elements $v\otimes t_i^k$ in the quotient, which we denote suggestively by $v_{[k]}^i$). One can define the action of $v_{[k]}^i$ on $w\in W^i$ by $v_{(k)}^{W^i}(w)$, the endomorphism that appears as Fourier coefficient of

$$Y^{W^i}(v,t_i) = \sum_{i\in\mathbb{Z}} v_{(k)}^{W^i} t_i^{-k-1},\ \text{ where }\ v_{(k)}^{W^i}\in\mathrm{End}(W^i).$$

This induces a diagonal action of $\bigoplus_{i=0}^n \mathfrak{L}_{P_i}(V)$ on the tensor product $W^\bullet := W^1\otimes\cdots\otimes W^n$.

To summarize: The sheaf of Chiral Lie algebras $\mathcal{L}_{\widehat{C}_{g,n}}$ on $\widehat{\mathcal{M}}_{g,n}$ acts on the constant bundle $W^\bullet\otimes\mathscr{O}_{\widehat{\mathcal{M}}_{g,n}}$ so that $\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V)\cdot(W^\bullet\otimes\mathscr{O}_{\widehat{\mathcal{M}}_{g,n}})$ is a sub-module of $W^\bullet\otimes\mathscr{O}_{\widehat{\mathcal{M}}_{g,n}}$. The sheaf is

$$\widehat{\mathbb{V}}_g(V;W^\bullet) = W^\bullet\otimes\mathscr{O}_{\widehat{\mathcal{M}}_{g,n}}\Big/\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V)\cdot(W^\bullet\otimes\mathscr{O}_{\widehat{\mathcal{M}}_{g,n}}). \tag{8}$$

3.1.2. *Descent.* We describe how the sheaf of coinvariants depicted in (8) can be shown to descend, first to $\overline{\mathcal{J}}_{g,n}$, which is the space of tuples $(C, P_\bullet, \tau_\bullet)$, where $\tau_\bullet = (\tau_1,\ldots,\tau_n)$ with τ_i a non-zero 1-jet of a formal coordinate at P_i for each i, and then when possible, to $\overline{\mathcal{M}}_{g,n}$:

$$\widehat{\mathcal{M}}_{g,n}\xrightarrow{\ \pi^1\ }\overline{\mathcal{J}}_{g,n}\xrightarrow{\ \pi^2\ }\overline{\mathcal{M}}_{g,n}\ .$$

Changing coordinates defines a transitive action of the group scheme $(\mathrm{Aut}\,\mathcal{O})^n$ (Lecture 1) on the fibers of the projection $\pi=\pi^2\circ\pi^1:\widehat{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,n}$. This action gives $\widehat{\mathcal{M}}_{g,n}$ the structure of a principal $(\mathrm{Aut}\,\mathcal{O})^n$-bundle over $\overline{\mathcal{M}}_{g,n}$. Moreover, $\mathrm{Aut}\,\mathcal{O}=\mathbb{G}_m\ltimes\mathrm{Aut}_+\mathcal{O}$, and the projection factors as a composition of an $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$-torsor and a $\mathbb{G}_m^{\oplus n}$-torsor (see §1.4 for definitions):

$$\begin{tikzcd} & \widehat{\mathcal{M}}_{g,n} \arrow[dl, "(\mathrm{Aut}_+\mathcal{O})^{\oplus n}"'] \arrow[dd, "(\mathrm{Aut}\,\mathcal{O})^{\oplus n}"] \\ \overline{\mathcal{J}}_{g,n} \arrow[dr, "\mathbb{G}_m^{\oplus n}"'] & \\ & \overline{\mathcal{M}}_{g,n}. \end{tikzcd} \tag{9}$$

For the descent to $\overline{\mathcal{J}}_{g,n}$, we note that the actions of $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$ and of $\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V)$ on $W^\bullet \otimes \mathscr{O}_{\widehat{\mathcal{M}}_{g,n}}$ are compatible [DGT22a]. In other words the action of $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$ on $W^\bullet \otimes \mathscr{O}_{\widehat{\mathcal{M}}_{g,n}}$ preserves the submodule $\mathcal{L}_{\widehat{C}_{g,n}\setminus P_\bullet}(V)(W^\bullet \otimes \mathscr{O}_{\widehat{\mathcal{M}}_{g,n}})$, inducing an action of $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$ on $\widehat{\mathbb{V}}_g(V; W^\bullet)$. Therefore, as explained in detail in [DGT22a], descending along this $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$-torsor, one obtains a sheaf of coinvariants $\mathbb{V}^J(V; M^\bullet)$ on $\overline{\mathcal{J}}_{g,n}$.

$$\mathbb{V}^J(V; M^\bullet) := \left((\pi^1)_* \, \widehat{\mathbb{V}}_g(V; W^\bullet)\right)^{\mathrm{Aut}_+\mathcal{O}^n}.$$

The descent to $\overline{\mathcal{M}}_{g,n}$ is more complicated. If the conformal dimensions are not integers, an argument as given above isn't immediately available. One must first use an idea inspired by Tsuchimoto [Tsu93], explained heuristically as follows. When conformal dimensions of modules are rational, there is a natural line bundle pulled back from $\overline{\mathcal{M}}_{g,n}$ such that when one tensors it with $\mathbb{V}^J(V; M^\bullet)$, the $\mathbb{G}_m^n$ action becomes compatible, and an analogous descent to $\overline{\mathcal{M}}_{g,n}$ is possible. One can then tensor back the resulting sheaf on $\overline{\mathcal{M}}_{g,n}$ with the dual of the original line bundle. The actual argument is more complicated, and uses root stacks, as described in [DGT22a, Remark 8.7.3, (ii)].

When the sheaf $\mathbb{V}(V, W^\bullet)$ is defined on $\overline{\mathcal{M}}_{g,n}$, then fibers at $(C, P_\bullet)$ are often denoted:

$$\mathbb{V}(V, W^\bullet)|_{(C,P_\bullet)} = [W^\bullet]_{\mathcal{L}_{C\setminus P_\bullet}} = \frac{W^1 \otimes \cdots \otimes W^n}{\mathcal{L}_{C\setminus P_\bullet}(V) \cdot (W^1 \otimes \cdots \otimes W^n)}.$$

Example 3.1. *Admissible modules over C_2-cofinite vertex operator algebras have rational conformal weights* [Miy04a, Corollary 5.10]*, so their sheaves of coinvariants descend to $\overline{\mathcal{M}}_{g,n}$. Many examples considered in the VOA literature are C_2-cofinite, including Affine VOAs $V_k(\mathfrak{g})$ and $L_k(\mathfrak{g})$ where $\mathfrak{g}$ is a simple Lie algebra, and $k \in \mathbb{Z}_{>0}$, Virasoro V_c (with c in the discrete series), lattice VOAs, Holomorphic VOAs (like the moonshine module). There are many otheres, for instance obtained as tensor products, orbifold algebras and quotients. See* [DGT21, DGT22a, DGT22b, DG21] *for specific examples and references.*

Example 3.2. *There are some non-C_2-cofinite VOAs whose modules are known to have rational conformal weights. For instance $L_k(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra and k is an admissible level that is not a positive integer, are not C_2-cofinite. However, by* [Ara16, Main Theorem]*, the conformal weights of modules in category $\mathcal{O}$ are rational, as they are from (slightly) larger categories (that allow for dense modules, spectral flow twists and finite length extensions as studied in for instance in* [CRW14]*) where the weights are determined by those in Arakawa's classification by formulas that preserve rationality.*

3.2. **Part 2: Properties.** We will next discuss various properties of sheaves of coinvariants and dual sheaves of conformal blocks, summarized in the following table:

V	$A = V/O(V)$	modules $W = \bigoplus_{d\in\mathbb{N}} W_d$	sheaves of coinvariants
C_1	finitely generated	finitely generated over W_0	quasi-coherent on $\overline{\mathcal{J}}_{g,n}$
C_2	finite dimensional	finitely generated over W_0 rational conformal weight	coherent on $\overline{\mathcal{M}}_{g,n}$
rational	finite dimensional and semi-simple	finitely generated over W_0 completely reducible	
rational and C_2	finite dimensional and semi-simple	finitely generated over W_0 rational conformal weight completely reducible	vector bundles on $\overline{\mathcal{M}}_{g,n}$
strongly rational	finite dimensional and semi-simple	finitely generated over W_0 rational conformal weight completely reducible	vector bundles on $\overline{\mathcal{M}}_{g,n}$ Chern classes tautological
affine	depends	depends	$g = 0$ globally generated $\&$ Chern classes free

It is possible to improve the diagram. For instance, certain categories of modules over C_1-cofinite (but not C_2-cofinite) VOAs have rational conformal weights and so define coinvariants on $\overline{\mathcal{M}}_{g,n}$. Moreover, by [DGK22], if V is C_1-cofinite, then fibers at nodal curves have factorization resolutions. Consequently, sheaves of coinvariants on $\overline{\mathcal{M}}_{g,n}$ are coherent if V is C_2-cofinite [DGK22]. If V is both rational and C_2-cofinite, factorization resolutions give factorization of coinvariants, and sheaves of coinvariants are locally free of finite rank (ie. are vector bundles).

The main results on the table above rely on factorization resolutions, which allows one to reinterpret vector spaces of coinvariants on nodal curves in terms of coinvariants on a curve with fewer singularities. To describe these, suppose C has only one node $Q \in C$, and let $\widetilde{C} \to C$ be the normalization of C, with points Q_+ and Q_- lying over Q. To define coinvariants on $\widetilde{C}$, one has the $A(V)$ module W^i for each point P_i, and it is natural to assign a *single* $A(V)$*-bimodule* E to the pair of markings Q_+ and Q_- of $\widetilde{C}$. One may then define the vector space of coinvariants $[W^\bullet \otimes \Phi(B)]_{(\widetilde{C},P_\bullet\cup Q_\pm)}$, where Φ is the functor, defined in§2, taking $A(V)$-bimodules to $\mathscr{U}(V)^2$-modules, and compatible with Zhu's functor $\mathcal{M}$, also defined in §2. Our main technical tool asserts that if V is C_1-cofinite (which implies that $A(V)$ is finitely generated), the coinvariants $[W^\bullet]_{(C,P_\bullet)}$ and $[W^\bullet \otimes \Phi(A(V))]_{(\widetilde{C},P_\bullet\cup Q_\pm)}$ coincide (here $A(V)$ is considered as a bimodule over itself).

On the one hand, this rephrases the coinvariants $[W^\bullet]_{(C,P_\bullet)}$ in terms of something associated to a less singular curve $\widetilde{C}$. However, the expression $[W^\bullet \otimes \Phi(A(V))]_{(\widetilde{C},P_\bullet\cup Q_\pm)}$ is, a priori, of a somewhat different nature, as we have associated a single bimodule as opposed to a pair of $A(V)$ modules to the points Q_+ and Q_-. On the other hand, when $A(V)$ is *factorizable*, that is, it can be written as a sum of the form $A(V) = \bigoplus(X_0^+ \otimes X_0^-)$, we may identify the spaces

$$[W^\bullet \otimes \Phi(A(V))]_{(\widetilde{C},P_\bullet\cup Q_\pm)} = \bigoplus [W^\bullet \otimes X^+ \otimes X^-]_{(\widetilde{C},P_\bullet\cup Q_\pm)}$$

with the vector space of coinvariants obtained by assigning the modules $W^\bullet$ to the points $P_\bullet$, and assigning $X^\pm$ to $Q_\pm$. In particular, if V is rational and C_2-cofinite (which implies that $A(V)$ is finite and semi-simple) then $A(V) = \bigoplus(X_0 \otimes X_0^\vee)$, a finite sum over all simple $A(V)$-modules X_0, and this recovers [DGT22a, Factorization Theorem]:

$$[W^\bullet]_{(C,P_\bullet)} \cong \bigoplus [W^\bullet \otimes X \otimes X^\vee]_{(\widetilde{C},P_\bullet \cup Q_\pm)}, \tag{10}$$

a finite sum, indexed by the isomorphism classes of all simple V-modules X.

It turns out that this is sharp—an associative algebra A is isomorphic to a finite direct sum of tensor products of left and right A-modules if and only if A is finite and semi-simple. It follows that for naturally occurring VOAs for which $A(V)$ is not semi-simple, but does satisfy finiteness conditions (such as being finitely generated or finite dimensional), other approaches are needed to relate coinvariants on nodal curves to coinvariants on curves with fewer singularities.

The strategy we take in [DGK22] is to observe that if V is C_1-cofinite, then $A(V)$ has what we call a factorization resolution $\cdots \xrightarrow{\alpha} \oplus (X_0 \otimes Y_0) \to A(V) \to 0$. We show that from any such factorization resolution of $A(V)$, one obtains a factorization presentation of nodal coinvariants. In particular, this expresses coinvariants at nodal curves as a quotient of a sum of coinvariants on the normalization (as in (10)). However in this case, the sum, which may not be finite, is indexed by indecomposable V-modules. This factorization presentation specializes to (10) if V is rational and C_2-cofinite, giving an alternative proof.

The proof involves two steps: First an application of the fact mentioned earlier, which asserts that if V is C_1-cofinite, one has a natural isomorphism

$$[W^\bullet]_{(C,P_\bullet)} \cong [W^\bullet \otimes \Phi(A(V))]_{(\widetilde{C},P_\bullet \cup Q_\pm)}. \tag{11}$$

In the second step, the right hand side of (11) shown to be the cokernel of a right exact functor applied to a factorization resolution of $A(V)$.

Add detail here

There are two consequences of such expression in case V is C_2-cofinite (which implies that $A(V)$ is finite dimensional). In this case $A(V)$ has a unique bimodule decomposition as a finite sum of principal indecomposable $A(V)$-bimodules (this has to be proved). We can then show:

$$[W^\bullet]_{(C,P_\bullet)} \cong \bigoplus [W^\bullet \otimes X \otimes X']_{(\widetilde{C},P_\bullet \cup Q_\pm)} \oplus \bigoplus [W^\bullet \otimes \Phi(I)]_{(\widetilde{C},P_\bullet \cup Q_\pm)}, \tag{12}$$

where X and X' are dual simple V-modules obtained by applying the functor Φ to specified simple indecomposable bimodules in its bimodule decomposition, and $\Phi(I)$ is a tensor product of indecomposable V-modules given by the remaining principal indecomposable bimodules I. This is illustrated in [DGK22] for the triplet $\mathcal{W}(p)$ and the Symplectic Fermions $F(d)^+$, important families of strongly finite, non-rational VOAs. One may refine (12) via a factorization resolution of the indecomposable bimodules I, as is demonstrated in [DGK22] for the $\mathcal{W}(p)$.

The second consequence is that sheaves of coinvariants defined by representations of C_2-cofinite VOAs are coherent on $\overline{\mathcal{M}}_{g,n}$. Vector spaces of coinvariants at smooth pointed coordinatized curves were shown to be finite dimensional in this generality in [DGT22a, Proposition 5.1.1], based on arguments made for a related construction in [AN03]. This result improves [DGT22a, Theorem 8.4.2.] which concludes that spaces of coinvariants at nodal curves are also finite dimensional if V is both C_2-cofinite and rational. This result achieves the first step towards showing that the sheaves we consider may form vector bundles (see Lecture 4 for a discussion of the problem).

A vertex operator algebra V is C_1 cofinite if and only if it is (strongly) generated in finite degree. There are strongly finitely generated vertex operator algebras which are not rational or C_2-cofinite, and in this case fibers of the sheaf coinvariants at a point $(C, P_\bullet)$ should be regarded as dependent on tangent data to the curve C at the marked points P_i. The affine VOAs $V_\ell(\mathfrak{g})$, and $L_\ell(\mathfrak{g})$ are defined for all $\ell \in \mathbb{C}$ with $\ell \neq -h^\vee$. They are generated in degree 1 (so are C_1-cofinite), but are C_2-cofinite if and only if ℓ is a positive integer. The Virasoro vertex operator algebras Vir_c for $c \in \mathbb{C}$ are generated in degree 2, but if c is not in the discrete series, they are not rational or C_2-cofinite. If the sums in the numerator are not finite for such examples, they cannot be used (as we do in the C_2-cofinite case), to prove that the nodal coinvariants are finite dimensional, but nevertheless these ideas apply.

4. LECTURE 4: OPEN PROBLEMS

There are a number of sufficient conditions, some we have discussed, such that if satisfied by a vertex algebra V and V-modules W^i will guarantee that the corresponding sheaves of coinvariants $\mathbb{V}(V, W^\bullet)$ have *good properties*. In this lecture I will present five open problems, the first three of which involve finding conditions necessary to ensure these good properties hold. The remaining problems aim to determine whether these sheaves give new information about the moduli space of curves, beyond those defined by affine VOAs, and whether this new information has anything to say about the F-Conjecture and the Mori Dream Space Conjecture, two longstanding problems.

Here is a summary of what we know:

$V = \oplus_{d\in\mathbb{N}} V_d$	$A = V/O(V)$	sheaves of coinvariants
CFT-type		connection and QC on $\star$
C_1	FG	factorization presentations
C_2	FD	defined on $\overline{\mathcal{M}}_{g,n}$, and C
R	FD & SS	
R & C_2	FD & SS	VB on $\overline{\mathcal{M}}_{g,n}$
SR	FD & SS	VB on $\overline{\mathcal{M}}_{g,n}$ tautological
affine	depends	$g=0$ GG & free classes

On $\star$ refers to the fact that the sheaves are defined on $\widehat{\mathcal{M}}_{g,n}$, $\overline{\mathcal{J}}_{g,n}$, and on $\overline{\mathcal{M}}_{g,n}$ if $\Delta_{W^i} \in \mathbb{Q}$. FG stands for finitely generated, FD for finite dimensional, SS means semi-simple, R rational, SR means strongly rational, Δ_W is the conformal weight of the module W, QC is quasi-coherent, C means coherent, VB stands for vector bundle, GG is globally generated. For definitions see §2.6.

These findings (and some other work) has led to the following questions:

- Let V be a C_1-cofinite VOA and $\{W^i\}_{i=1}^n$ be n simple admissible V modules. What conditions must hold on $(V, W^\bullet)$ so that sheaves $\mathbb{V}(V, W^\bullet)$:
  (1) are vector bundles?
  (2) have tautological Chern classes?
  (3) are positive? For instance:
    (a) are globally generated (so Chern classes are base point free);
    (b) Chern classes are nef but not free?
    (c) have effective Chern classes, even if not nef?
- Do sheaves give any new information about the moduli space of curves? For instance:
  (4) Do sheaves of coinvariants define *new* nef classes?
  (5) Can one use the classes from these sheaves of coinvariants to say something new about the F-Conjecture or the Mori Dream Space Question

These questions will be discussed in more detail next.

That sheaves of coinvariants are vector bundles for any choice of modules in some category, is equivalent to that category giving a (particular type of) conformal field theory. For instance, if V is rational and C_2-cofinite, any choice of admissible modules forms a vector bundle, and the category of admissible modules forms a rational conformal field theory. If V is strongly rational, then Chern characters form a cohomological field theory [DGT22b]. An important open problem, discussed in Lecture 4, is whether or not the coherent sheaves defined by modules over C_2-cofinite V are also locally free. The categories of modules over strongly finite VOAs are believed to correspond to logarithmic conformal field theories. Unlike for vector bundles defined by admissible modules over strongly rational VOAs, it is unlikely that Chern classes of such vector bundles defined by modules over strongly finite V would be tautological nor that Chern characters would form CohFTs.

4.1. **Questions:** (a) Given $W^1, \ldots, W^n$ modules over a C_2-cofinite VOA V, is $\mathbb{V}(V; W^\bullet)$ a vector bundle? (b) If not, what additional assumptions are necessary so it is?

The question of whether sheaves of coinvariants defined by representations of strongly finite VOAs are vector bundles was discussed in [DGK22], where we started to build the infrastructure which may help answer this question.

4.1.1. *Motivation.* Following [DGK22], to motivate the question, we briefly describe, besides what is summarized in the table above, the important properties these sheaves have, lending to a comparison with the strongly rational case. For instance:

(A) V-modules are objects of a modular tensor category (MTC) [HL95, Hua05].
(B) In the language of [MS89], rational conformal field theories (RCFTs) are determined by a coherent sheaf of coinvariants (and dual sheaf of conformal blocks), and of [FS87], by vector bundles of coinvariants together with their projectively flat connection.
(C) Properties of the MTC from (A) correspond to those of sheaves of (B) [BK01].

A modular tensor category is a braided tensor category with additional structure (see [Tur94], and in this context [CG17, §2.6]). If V is strongly rational, then every V-module is ordinary, and can be expressed as a finite sum of simple V-modules S^i. By [Zhu94], fusion coefficients

$$W^i \otimes W^j = \mathcal{N}_{ij}^k \, W^k, \tag{13}$$

are determined by the dimensions of vector spaces of conformal blocks on $\overline{\mathcal{M}}_{0,3}$

$$\mathcal{N}_{ij}^k = \dim \left(\mathbb{V}_0(V; (W^i, W^j, (W^k)')^\vee \right) \in \mathbb{Z}_{\geq 0}.$$

Equation (13) gives the product structure on the fusion ring $\mathrm{Fus}^{Simp}(V) = \mathrm{Span}_{\mathbb{Z}}\{S^i\}$ spanned by (isomorphism classes) of simple V-modules (with unit element V).

A C_2-cofinite but not rational VOA has at least one indecomposable but not simple module (such a VOA or conformal field theory is called logarithmic). Let V be strongly finite (so as in 2.6, V is C_2-cofinite, simple, and self-dual).

(A') The braided tensor category $Mod^{gr}(V)$ [HLZ14] is conjecturally log-modular [CG17].

(B') Logarithmic conformal field theories (LCFTs) are determined by finite dimensional vector spaces of coinvariants and conformal blocks defined by V-modules [CG17].

(C') Features of (A') and (B') correspond to properties of the sheaves $\mathbb{V}(V; W^\bullet)$.

Log modular categories are braided tensor categories with certain additional structure (see [CG17, Def 3.1]). If V is strongly finite, the important modules are the (finitely many) projectives $\{P_i\}_{i\in I}$, which consist of the simple modules and their projective covers. By [CG17, Prop 3.2, part (d)], if [CG17, Conj 3.2] holds, then these projective V-modules form an ideal in $Mod^{g.r}(V)$, closed under tensor products, and taking contragredients, with fusion coefficients given in terms of dimensions of vector spaces of conformal blocks.

4.1.2. *Approach.* By [DGK22, Corollary], if V is C_2 cofinite, then $\mathbb{V}(V; W^\bullet)$ is coherent. Therefore, part (a) asks whether $\mathbb{V}(V; W^\bullet)$ is locally free. For V rational and C_2-cofinite, that $\mathbb{V}(V, W^\bullet)$ is locally free follows in large part from the sewing theorem [DGT22a, Theorem 8.5.1] (see [DGT22a, VB Corollary]). As we discuss in [DGK22] sewing theorems come up in a number of different contexts. According to the physical arguments described by Moore and Seiberg in [MS89], to construct a conformal field theory based on representations of VOAs, a key consistency condition is the modular invariance of the characters of irreducible representations of V. In [FS17], in studying CFTs, the authors are interested in the identification of spaces of correlation functions, which are compatible with the sewing of Riemann surfaces. Such conditions often include compatibility with respect to the sewing of surfaces (see [FS17] and references therein). A sewing result has been proved for curves of genus $g \in \{0, 1\}$ by Huang in [HL13]. In [Zhu96], this was shown for strongly rational VOAs, and in [Miy04b] an analogous statement was shown in case V is strongly finite.

4.2. **Questions:** (a) What are the Chern classes of $\mathbb{V}(V; W^\bullet)$? (b) Are they tautological?

In case V is strongly rational, then by [DGT22b, Theorem 1], the collection consisting of the Chern characters of all vector bundles of coinvariants forms a semisimple cohomological field theory, giving rise to explicit expressions for Chern classes (see [DGT22b, Corollaries 1 and 2]). This was proved following the original result for Verlinde bundles [MOP15, MOP$^+$17].

If V is strongly finite, but not rational, while one still has Chern characters (with $\mathbb{Q}$-coefficients) an analogous CohFT is not obviously available. For a semisimple CohFT, one naturally obtains the structure of a Fusion ring, which is necessarily semi-simple. In the strongly finite, non-rational case, there are three options for what could play the role of a fusion ring, including $\mathrm{Fus}^{Simp}(V)$ spanned over $\mathbb{Z}$ by the projective modules. For example the $4p - 2$ simple and indecomposable $\mathcal{W}(p)$-modules discussed in §2.4 are closed under tensor products and their $\mathbb{Z}$-span forms $\mathrm{Fus}^{Simp}(\mathcal{W}(p))$, but this ring is not semi-simple. One may therefore need new ideas for computing Chern classes. As factorization presentations are quotients, it seems unlikely that classes will be generally tautological.

4.3. **Question: Which sheaves of coinvariants have positivity properties?** The strongest positivity one can hope for is that sheaves of coinvariants are globally generated, and this question has been considered in [DG21], with the best known results for affine VOAs, constructed from (certain quotients of) affinizations of a Lie algebra $\mathfrak{g}$, and $\ell \in \mathbb{C}$, with $-\ell$ not equal to the dual Coxeter number. We next summarize the results from [DG21], which were inspired by results from [Fak12]. For $\mathfrak{g}$ simple, the simple affine VOA $V = L_\ell(\mathfrak{g})$ is of CFT-type, generated by its degree 1 component $V_1 \cong \mathfrak{g}$. It is rational and C_2-cofinite if and only if $\ell \in \mathbb{Z}_{>0}$. Sheaves $\mathbb{V}(L_\ell(\mathfrak{g}), W^\bullet)$, were shown to be vector bundles in [TUY89], and globally generated on $\overline{\mathcal{M}}_{0,n}$ in [TUY89, Fak12].

By [GG12, Theorem 1], sheaves of coinvariants defined by n simple admissible modules over a vertex operator algebra strongly generated in degree 1 and of CFT-type, are globally generated on $\overline{\mathcal{J}}_{0,n}$, and on $\overline{\mathcal{M}}_{0,n}$, if they descend.

As in [DGK22, Remark 1.4.1], by [Lia94] the VOAs in [DGK22, Theorem 1] are quotients of the affinization of a not necessarily reductive Lie algebra structure on their degree 1 component. By [DM06], if V is simple, strongly generated in degree one, rational, and self-contragredient, then $V \cong \bigotimes_{i=1}^r L_{\ell_i}(\mathfrak{g}_i)$, with $\mathfrak{g}_i$ simple Lie algebras, $\ell_i \in \mathbb{Z}_{>0}$, and $V_1 \cong \bigoplus_{i=1}^r \mathfrak{g}_i$. In [DGK22, Theorem 1], V need not be simple, C_2-cofinite or rational, and may for instance be applied to $L_\ell(\mathfrak{g})$, for $\mathfrak{g}$ simple and ℓ admissible, and not in $\mathbb{Z}_{>0}$. Such VOAs $L_\ell(\mathfrak{g})$ are not C_2-cofinite, but are quasi-lisse, a natural generalization of C_2-cofiniteness, introduced in [AK18]. It follows from [Ara16, Main Theorem], that simple admissible highest weight modules have rational conformal weights, as do more general V-modules (see Remark 3.2). As in [Lia94], there are many other examples to which Theorem 1 applies.

By [GG12, Corollary 1], sheaves described in Theorem 1 are coherent. This improves [AN03] giving coherence of $\mathbb{V}(V, W^\bullet)$ on $\mathcal{M}_{0,n}$ for V is C_2-cofinite, self-contragredient, and of CFT-type, [DGK22] proving coherence on $\overline{\mathcal{M}}_{0,n}$ for C_2 cofinite V of CFT-type, and [DGT22a] where one assumes V is also rational. Such sheaves are vector bundles on $\mathcal{M}_{0,n}$. If V is C_2-cofinite and rational, by [DGT22a] these are vector bundles on $\overline{\mathcal{M}}_{g,n}$, these vector bundles are globally generated on $\overline{\mathcal{M}}_{0,n}$, extending [TUY89, Fak12].

While we haven't found conditions to guarantee global generation for $g > 0$, or for VOAs which are not strongly generated in degree 1, to illustrate the subtlety of this problem, we give several representative examples, including:

- globally generated and positive bundles from VOAs not strongly generated in degree 1;
- and sheaves of coinvariants that are not globally generated.

Let X be a projective, not necessarily smooth variety defined over an algebraically closed field. Good references for the concepts below are [Laz04a, Laz04b].

Definition 4.1. *A variety X is called $\mathbb{Q}$-factorial if every Weil divisor on X is $\mathbb{Q}$-Cartier. We assume today that X is a $\mathbb{Q}$-factorial normal, projective variety over the complex numbers. The moduli spaces $\overline{\mathrm{M}}_{g,n}$ have these properties.*

Definition 4.2. *Divisors D_1 and D_2 are numerically equivalent, written $D_1 \equiv D_2$, if they intersect all irreducible curves in the same degree. Curves C_1 and C_2 are numerically equivalent, written $C_1 \equiv C_2$ if $C_1 \cdot D = C_2 \cdot D$ for every irreducible subvariety D of codimension one in X.*

Definition 4.3. *We set $N_1(X)_{\mathbb{Z}}$ equal to the vector space of curves up to numerical equivalence, and $N^1(X)_{\mathbb{Z}}$ equal to the vector space of divisors up to numerical equivalence, and set*

$$\mathrm{N}^1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \;\; \mathrm{N}^1(X) = \mathrm{N}^1(X)_{\mathbb{R}} = N^1(X)_{\boldsymbol{Z}} \otimes_{\mathbb{Z}} \mathbb{R},$$

and

$$\mathrm{N}_1(X)_{\boldsymbol{Q}} = N_1(X)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \mathbb{Q}, \;\; \mathrm{N}_1(X) = \mathrm{N}_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Definition 4.4. *The pseudo effective cone $\overline{\mathrm{Eff}}_k(X) \subset \mathrm{N}_k(\overline{\mathrm{M}}_{g,n})$ is defined to be the closure of the cone generated by k-cycles with nonnegative coefficients. Similarly $\overline{\mathrm{Eff}}^k(X) \subset \mathrm{N}^k(X)$ is defined to be the closure of the cone generated by cycles of codimension k with nonnegative coefficients.*

The cones $\overline{\mathrm{Eff}}_k(X)$, and $\overline{\mathrm{Eff}}^k(X)$ are full dimensional, spanning the vector spaces $\mathrm{N}_k(X)$, and $\mathrm{N}^k(X)$. They are pointed (containing no lines), closed, and convex.

Definition 4.5. *The Nef Cone $\mathrm{Nef}^k(X) \subset \mathrm{N}^k(X)$ is the cone dual to $\overline{\mathrm{Eff}}_k(X)$.*

As the dual of $\overline{\mathrm{Eff}}_k(X)$, the nef cone has all of the nice properties that $\overline{\mathrm{Eff}}_k(X)$ does.

The nef cone can also be defined as the closure of the cone generated by semi-ample divisors – divisors that correspond to morphisms, and

$$f : X \to Y \text{ is a regular map, then } f^*(\mathrm{Nef}(Y)) \subset \mathrm{Nef}^1(X).$$

Given a projective variety Y, and a morphism $f : X \longrightarrow Y \hookrightarrow \mathbb{P}^N$, then for any ample divisor $A = \mathcal{O}(1)|_Y$ on Y, one has the pullback divisor $D = f^*A$ on X is base point free. In fact, this divisor D is not only base point free, it has the much weaker property that it is nef. For if C is a curve on our projective variety X, then by the projection formula

$$D \cdot C = f_*(D \cdot C) = A \cdot f_*C,$$

which is zero if the map f *contracts* C, and otherwise, as A is ample, it is positive.

It is not true that every nef divisor on an arbitrary proper variety X has an associated morphism; To have such a property would be very special (a dream situation). But as we saw above, the divisors that give rise to maps do live in the nef cone, and for that reason the nef cone can be used a tool to understand the birational geometry of the space.

The following is an even more refined concept that won't be mentioned in the lecture.

Definition 4.6. *For a $\mathbb{Q}$-Cartier divisor D on a proper variety X, we define:*

- *the stable base locus of D to be the union (with reduced structure) of all points in X which are in the base locus of the linear series $|nmD|$, for all n, where m is the smallest integer ≥ 1 such that mD is Cartier;*

- *A moving $\mathbb{Q}$-Cartier divisor to be a divisor whose stable base locus has codimension 2 or more; and*
- *the moving cone* $\mathrm{Mov}(X)$ *of* X*, is the closure of the cone of moving divisors.*

Sufficiently high and divisible multiples of any effective divisor D on X will define a rational map (although not necessarily a morphism) from X to a projective variety Y. The stable base locus of D is the locus where the associated rational map will not be defined. The pseudo-effective cone may be divided into chambers having to do with the stable base loci [ELM$^+$06, ELM$^+$09]. Moreover, if

$$f : X \dashrightarrow Y \text{ is a rational map, then } f^*(\mathrm{Nef}(Y)) \subset \mathrm{Mov}(X),$$

and we have

$$\mathrm{Nef}^1(X) \subseteq \mathrm{Mov}(X) \subseteq \overline{\mathrm{Eff}}^1(X).$$

4.3.1. *Examples.* Next we consider a simple example to illustrate how even very crude information about the location of the cone of nef divisors with respect to the effective cone tells us valuable information about the geometry of the variety X, as we see for $\overline{\mathrm{M}}_g$.

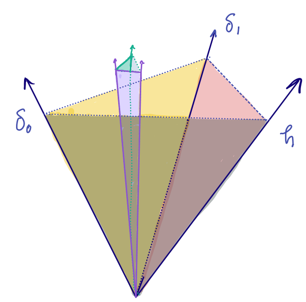


FIGURE 6. $\mathrm{Nef}^1(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{Eff}}^1(\overline{\mathrm{M}}_3)$ with generators λ, $12\lambda - \delta_0$, and $10\lambda - \delta_0 - 2\delta_1$.

In general we can say the following:

Theorem 4.7. *Every nef divisor on* $\overline{\mathrm{M}}_g$ *is big. In particular, there are no morphisms, with connected fibers from* $\overline{\mathrm{M}}_g$ *to any lower dimensional projective varieties other than a point.*

Theorem 4.7 says that the nef cone of $\overline{\mathrm{M}}_g$ sits properly inside of the cone of effective divisors– and their extremal faces only touch at the origin of the Nerón Severi space.

The statement for pointed curves is a little bit more complicated, but still very simple in the grand scheme of things:

Theorem 4.8. *For* $g \geq 2$*, any nef divisor is either big or is numerically equivalent to the pullback of a big divisor by composition of projection morphisms. In particular, for* $g \geq 2$*, the only morphisms with connected fibers from* $\overline{\mathrm{M}}_{g,n}$ *to lower dimensional projective varieties are compositions of projections given by dropping points, followed by birational maps.*

4.4. **Do sheaves of coinvariants give new nef classes?** We wonder if first Chern classes of sheaves reside outside of the cone spanned by first Chern classes of sheaves defined by affine VOAs $L_\ell(\mathfrak{g})$ where $\mathfrak{g}$ is a simple Lie algebra and $\ell \in \mathbb{Z}_{>0}$?

4.5. **F-Conjecture and MDS Conjecture.** The F-Conjecture and MDS Conjecture arise from the general observation that $\overline{\mathcal{M}}_{g,n}$ resembles other very well understood spaces. For instance, as we have seen as a moduli space, $\overline{\mathcal{M}}_{g,n}$ can be compared to a Grassmannian variety (and Mumford did this when he defined the tautological ring), and as Kapranov proved, $\overline{\mathcal{M}}_{0,n}$ is a quotient of a Grassmannian. As Fulton pointed out by, the action of the symmetric group S_n on $\overline{\mathcal{M}}_{g,n}$ by permuting the marked points, can be compared with the action of an algebraic torus $G \cong (\mathbb{C}^*)^n$ on a toric variety, or the transitive action of an algebraic group G a homogeneous variety. In fact S_n is the automorporphism group of $\overline{\mathcal{M}}_{0,n}$, as Fulton predicted it was.

Group actions are useful. For instance those sub-loci of a toric or homogeneous variety that are preserved by the group action play an important role in understanding their cycle structure.

An effective cycle E of dimension k on a variety X of dimension d is a formal sum of numerical equivalence classes of k-dimensional sub-loci on X. Two effective cycles E_1 and E_2 are numerically equivalent, written $E_1 \equiv E_2$, if the number of points (counted with multiplicity) of the intersections $E_1 \cap Z$ and $E_2 \cap Z$ are equal, for all complementary sub-loci $Z \subset X$ of dimension $d-k$. There are other (related) types of equivalence including algebraic and linear.

Since sums and positive multiples of effective cycles remain effective, these form cones, which for proper varieties live in finite dimensional vector spaces. These cones (and their closures) are combinatorial devices that encode geometric data about proper varieties. On (complete) toric varieties and on homogeneous varieties, on which a group G acts transitively, the G-invariant loci determine such cones. Boundary cycles (equivalence classes of boundary loci) are analogous to G-invariant loci on a homogeneous or toric variety. It is natural therefore to ask, by analogy, whether the boundary loci on $\overline{\mathcal{M}}_{g,n}$ play the same important role.

4.5.1. *The F-Conjecture.* Recall from the first lecture that in $\overline{\mathrm{M}}_{g,n}$, the locus

$$\delta^k(\overline{\mathrm{M}}_{g,n}) = \{(C, \vec{p}) \in \overline{\mathrm{M}}_{g,n} \ : \ C \text{ has at least k nodes } \}$$

has codimension k. For each k, the set $\delta^k(\overline{\mathrm{M}}_{g,n})$ decomposes into irreducible component indexed by dual graphs Γ with k edges. Moreover, the closure of the component corresponding to Γ contains components consisting of curves whose corresponding dual graph Γ' contracts to Γ. The resulting stratification of the space is both reminiscent and analogous to the combinatorial structure determined by the torus invariant loci of a toric variety.

On a complete toric variety, every effective cycle of dimension k can be expressed as a linear combination of torus invariant cycles of dimension k. Fulton compared the action of the symmetric group S_n on $\overline{\mathrm{M}}_{0,n}$ with the action of an algebraic torus a toric variety. Following this analogy, he asked whether a variety of dimension k could be expressed as an effective combination of boundary cycles of that dimension. As $\overline{\mathrm{M}}_{0,n}$ is rational, of dimension $n-3$, this is true for points and cycles of codimension $n-3$. For the statement to be true for divisors, it would say that every effective divisor would be in the cone spanned by the boundary divisors. This was proved false by Keel [GKM02, page 4] and Vermeire [Ver02], who found effective divisors not in the convex hull of the boundary divisors. For the statement to be true for curves, it would say that the Mori cone of curves is spanned by irreducible components of $\delta^{n-4}(\overline{\mathrm{M}}_{0,n})$: whose dual graph is distinctive: the only vertex that isn't trivalent has valency four. In particular, these are all curves that can be described as images of attaching or clutching maps from $\overline{\mathrm{M}}_{0,4}$.

Of course this question could just as well be asked for higher genus, and Faber did this, proving the statement for $\overline{\mathrm{M}}_3$ and $\overline{\mathrm{M}}_4$ (see eg. [Fab90, Intermezzo]).

In honor of Faber and Fulton, the numerical equivalence classes of the irreducible components of $\delta^{3g-4+n}(\overline{\mathrm{M}}_{g,n})$ are called F-Curves. One can ask the following question:

Question 4.9. *(The* F*-Conjecture* [GKM02]*) Is every effective curve numerically equivalent to an effective combination of* F*-Curves? Otherwise said, can one say that a divisor is nef, if and only if it nonnegatively intersects all the* F*-Curves?*

In [GKM02], we showed that in fact a positive solution to this question for S_g-invariant nef divisors on $\overline{\mathrm{M}}_{0,g+n}$ would give a positive answer for divisors on $\overline{\mathrm{M}}_{g,n}$. In particular, the birational geometry of $\overline{\mathrm{M}}_{0,g}$ controls aspects of the birational geometry of $\overline{\mathrm{M}}_g$. We know now that the answer to this question is true on $\overline{\mathrm{M}}_{0,n}$ for $n \leq 7$ [KM13], and on $\overline{\mathrm{M}}_g$ for $g \leq 24$ [Gib09].

4.5.2. *The MDS question.* Another analogy between $\overline{\mathrm{M}}_{0,n}$ and toric varieties prompted Hu and Keel to ask whether $\overline{\mathrm{M}}_{0,n}$ is a so-called Mori Dream Space. We now know, due to the work of Castravet and Tevelev, that this is not true in general. I'll define a Mori Dream Space and state the results of Castravet and Tevelev. To do so, we need first the definition of a so-called *small $\mathbb{Q}$-factorial modification* of X, defined as follows:

Definition 4.10. *Let X be a normal projective variety. A small $\mathbb{Q}$-factorial modification of X is a birational map*[4] *$f : X \to Y$ that is an isomorphism in codimension one (ie. is* small*) to a normal $\mathbb{Q}$-factorial projective variety Y. We refer to f as an SQM for short.*

Definition 4.11. *A normal projective variety X is called an MDS if:*

(1) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathrm{Q}}$;
(2) $\mathrm{Nef}(X)$ is generated by finitely many semi-ample line bundles;
(3) there is a finite collection of $SQMs$ $f_i : X \to X_i$ such that each X_i satisfies (1) and (2) and $\mathrm{Mov}(X)$ is the union of $f_i^*(Nef(X_i))$.

Extremely well behaved schemes, like toric and log Fano varieties, where the minimal model program can be carried out without issue, were deemed "Mori Dream Spaces" by Hu and Keel (MDS for short). The moduli space of stable n-pointed genus zero curves $\overline{\mathrm{M}}_{0,n}$ is Fano for $n \leq 6$, and so is a MDS in that range. While not Fano for $n \geq 7$, a comparison between the stratification

[4] In particular, this map f need not be regular.

of $\overline{\mathrm{M}}_{0,n}$, given by curves according to topological type, to the stratification of a toric variety given by its torus invariant sub-loci, prompted Hu and Keel to ask whether $\overline{\mathrm{M}}_{0,n}$ is a MDS for all n. This question has resulted in a great deal of work in the literature both about $\overline{\mathrm{M}}_{0,n}$ and related spaces. As Castravet and Tevelev point out in their paper, for about 15 years now, many researchers have tried to understand this particular problem. Other related questions go back to the work of Mumford.

Castravet and Tevelev in [CT15], prove that $\overline{\mathrm{M}}_{0,n}$ is not a MDS as long as n is at least 134. The authors assert that rather than compare $\overline{\mathrm{M}}_{0,n}$ to a toric variety, one should rather think of it as the blow up of a toric variety – namely, the blow up of the Losev Manin space $\overline{LM}_n$ at *the identity of the torus*. Using their work, in [GK16], González and Karu showed $\overline{\mathrm{M}}_{0,n}$ is not an MDS as long as n is at least 13. A very recent preprint of Hausen, Keicher, and Laface [HKL18] studies the blow-up of a weighted projective plane at a general point, giving criteria and algorithms for testing if the result is a Mori dream space. As an application, using the framework of Castravet and Tevelev, they show that $\overline{\mathrm{M}}_{0,n}$ is not an MDS as long as $n \geq 10$. The three cases 7, 8, and 9 therefore seem to remain open, as far as I know.

4.6. **What comes out of these questions?** In Castravet and Tevelev's proof that $\overline{\mathrm{M}}_{0,n}$ is not a MDS, they ultimately show that the third criterion of the definition for a MDS (see Definition 4.11) fails. If the second condition in the definition for a MDS, the prediction is that the Nef cone of $\overline{\mathrm{M}}_{0,n}$ should have a finite number of extremal rays, and that every nef divisor should be semi-ample. Moreover, if in the increasingly unlikely event that the F-Conjecture were to hold for $\overline{\mathrm{M}}_{0,n}$, then the Nef cone would have finitely many extremal rays. Therefore, it makes sense to ask:

Question 4.12. *(1) Is* $\mathrm{Nef}^1(\overline{\mathrm{M}}_{0,n})$ *polyhedral?*
(2) Is every element of $\mathrm{Nef}^1(\overline{\mathrm{M}}_{0,n})$ *semi-ample?*

It would be interesting to see that the answer to part (b) is yes, but that there are so many nef divisors that the answer to part (a) is no. This has led me to my current work about sheaves on $\overline{\mathrm{M}}_{g,n}$ defined by representations of vertex operator algebras.

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